Math 142 Group Projects

For this project you must work in groups of at most 3 students. You and your group will have to pick a topic by Friday, April 15. No two groups may present the same topic and the topics are awarded on a first come first serve basis. Each group will have 10-12 minutes to present their topic on Friday, April 22 or Monday, April 25. Every member of the group must speak during the presentation and be ready to answer questions.

Leonhard Euler and Euler's Formula.

1. Define the complex number i.

2. Explain Euler's formula and show how it can be derived using the Maclaurin series for sine, cosine, and e^x.

3. Show that Euler's formula immediately implies Euler's identity $e^{i\pi} + 1 = 0$ which relates the five most important numbers in all of mathematics.

4. Introduce the polar form for complex numbers and describe how it is related to Euler's formula. Show how to convert any complex number into polar form and show how to multiply two complex numbers in polar form (both algebraically and geometrically on the complex plane).

Archimedes and the quadrature of the parabola.

1. Give a description of how Archimedes was able to compute the area inside of a parabola without using integral calculus. Be sure to include details on how he added up an infinite geometric series, and where the terms in the series come from.

2. Use integral calculus to find the same area that Archimedes found.

e is irrational. Our goal is to show that e is an irrational number. To do so we use a proof technique called a *proof by contradiction*. We assume that e is rational and show that this leads to a contradiction. Thus we conclude that e is irrational.

If e were rational, then it would be a fraction $e = \frac{m}{n}$ where both m and n are integers and $n \geq 2$. By Taylor's theorem

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{e^z}{(n+1)!} = S_n + R_n$$

where

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

is the nth partial sum for e and

$$R_n = \frac{e^z}{(n+1)!} \quad \text{(for some } z \text{ between 0 and 1)}$$

is the remainder (see Taylor's theorem).

1. Show that $n!(\frac{m}{n} - S_n)$ must be an integer.

2. Show that $0 < n!R_n < 1$.

3. Show that $n!(\frac{m}{n} - S_n) = n!R_n$.

4. Explain why this is a contradiction. What does this contradiction force us to conclude?

Buffon's needle. This is a way to estimate π by tossing a needle on a floor between two straight lines. For this project, you will need to read about Buffon's needle and explain how you would use it to approximate π. Make sure you explain how integral calculus can be used to find the probabilities. You may want to use an applet such as the one at

> http://www.math.uah.edu/stat/applets/BuffonNeedleExperiment.html

to demonstrate how Buffon's needle works.

The gamma function. At first glance it appears that there is no real analog to the factorial function $n!$. In fact there is. It is a famous function called the *gamma function*. The gamma function has the form $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$.

1. Find $\Gamma(1)$ and $\Gamma(2)$.

2. Use WolframAlpha or a similar tool to sketch a graph of the gamma function.

3. Use integration by parts to show that that $\Gamma(x) = (x-1)\Gamma(x-1)$.

4. Combine your solutions to parts 1 & 2 to find $\Gamma(3)$. Then find $\Gamma(4)$ and $\Gamma(5)$.

5. Use mathematical induction to conclude that $\Gamma(n) = (n+1)!$

Euler and the infinitude of primes. Euler devised an amazing proof that there are infinitely many prime numbers by considering the harmonic series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

He observed that the harmonic series must be equal to the product

$$\prod_p \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right)$$

taken over all prime numbers p. (Here the $\prod$) symbol works just like the $\sum$ symbol, except you multiply instead of add).

1. Try expanding the first few terms (say when $p = 2, 3$, and 5) of the product above to get a feel for how it works.

2. Use the Fundamental Theorem of Arithmetic to explain why the product formula above contains every single fraction in the harmonic series exactly once.

3. Explain why $\sum_{i=1}^\infty 1/p^n$ is finite for every prime number p.

4. Now explain why Euler could conclude that there must be infinitely many primes.

Estimating π. There have been many approximations of π throughout history. One popular one is $\frac{22}{7}$.

1. Use polynomial long division and integration to show that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx = \frac{22}{7} - \pi$.

2. Explain why the integral above implies that $\frac{22}{7}$ must be greater than π.

3. How did Archimedes use a hexagon (actually two hexagons) to approximate π?

4. How did Archimedes improve on his hexagon estimate?

5. Compare the two methods for showing that $\pi < \frac{22}{7}$.