Calculus II - Math 142 **Old Final Exam Solutions**

I have posted hints for some of the problems and solutions for others. Make sure you understand the solutions, and understand how to use the hints, and why they were given.

1. (8 points) Evaluate the following integrals.

   (a) $\int e^x \cos(e^x)\, dx$.

   Solution: Use the u-substitution $u = e^x$.

   (b) $\int \tan^5 \theta \sec^2 \theta \, d\theta$.

   Solution: Use the u-substitution $u = \tan \theta$.

2. (8 points) Find the third degree Taylor polynomial for $f(x) = x^3 + 2x - 3$ centered at $c = 2$.

Solution: Use the table below to find the coefficients:

$f(x) = x^3 + 2x - 3$	$f(2) = 9$
$f'(x) = 3x^2 + 2$	$f'(2) = 14$
$f''(x) = 6x$	$f''(2) = 12$
$f^{(3)}(x) = 6$	$f^{(3)}(2) = 6$

So

$$P_3(x) = 9 + 14(x-2) + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3 =$$
$$= 9 + 14(x-2) + 6(x-2)^2 + (x-2)^3.$$

3. (8 points) Solve the differential equation.

$$\frac{dy}{dx} = \frac{\cos x}{y^2}$$

Solution: Separate variables to get:

$$y^2\,dy = \cos x\,dx.$$

Then integrate

$$\int y^2\,dy = \int \cos x\,dx$$
$$\frac{y^3}{3} = \sin x + C$$
$$y = \sqrt[3]{3\sin x + C}.$$

4. (8 points) For each of the following series, state whether it converges or diverges and give your reasoning.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1}}{6^n}$

Solution: This is a geometric series, so you can tell whether it converges by finding the common ratio. It is also an alternating series, so you could use the alternating series test.

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{k-1}$

Solution: Diverges by comparison with the harmonic series.

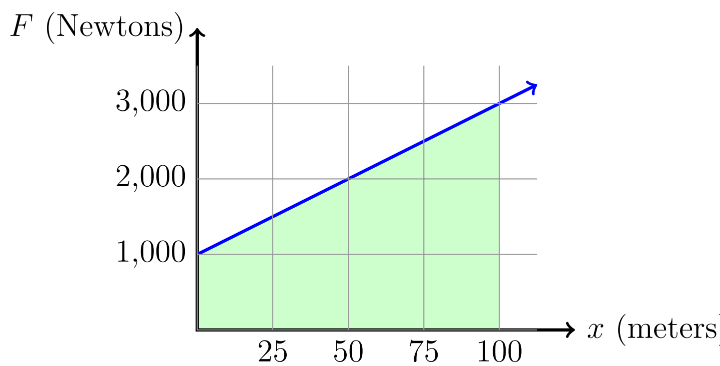
(c) $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi)$

Solution: The terms of this series alternate between $+1$ and -1. Since the terms don't converge to zero, the series cannot converge by the divergence test.

5. (4 points) Is the function $y = x^2 - 4x + 5$ invertible? If so, find the inverse. If not, explain why not.

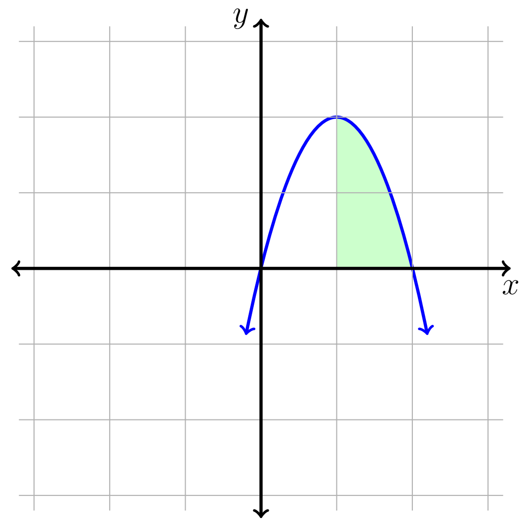
Solution: Not invertible because the graph is a parabola which fails the horizontal line test.

6. (4 points) Suppose I am pushing a heavy object over snow covered ground. The further I go, the deeper the snow gets, making me use more and more force to push the object. If the force I use as I push the object 100 meters is shown in the graph below, find the amount of work I did.



Solution: Work is $\int F\,dx$ which is just the area under this curve. Break the area into a rectangle plus a triangle (area of a triangle is $\frac{1}{2}$ base $\times$ height).

7. (8 points) Let $\mathcal{R}$ be the region under the curve $y = 4x - 2x^2$ from $x = 1$ to $x = 2$.



(a) Find the volume of the solid formed by revolving $\mathcal{R}$ around the y-axis.

Solution: $V = \dfrac{11\pi}{3}$.

(b) Setup, but do not evaluate an integral for the volume of the solid formed by revolving $\mathcal{R}$ around the x-axis.

Solution:

$$V = \int_1^2 \pi(4x - 2x^2)^2 \, dx.$$

8. (8 points) Suppose that $f(x) = \sin(x^3)$.

(a) Find a Maclaurin series for $f(x)$.

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}.$$

(b) Use part (a) to find an infinite series for the integral $\int_0^1 \sin(x^3)\, dx$.

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+4)(2n+1)!}.$$

9. (8 points) Evaluate the following integrals.

(a) $\int x^4 \ln x \, dx$

Solution: Tabular won't work since you can't easily integrate $\ln x$. Use integration by parts with

$$u = \ln x, \qquad dv = x^4 \, dx.$$

(b) $\int \frac{x^3+4}{x^2-4} \, dx$

Solution: Use polynomial long-division first. You get:

$$\int \frac{x^3+4}{x^2-4} \, dx = \int x + \frac{4x+4}{x^2-4} \, dx =$$

$$\frac{x^2}{2} + \int \frac{4x+4}{(x-2)(x+2)} \, dx =$$

Using partial fractions

$$\frac{4x+4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

so $A(x+2) + B(x-2) = 4x+4$. Plug-in $x = 2$, and $4A = 12$ so $A = 3$. Plug-in $x = -2$, and $-4B = -4$ so $B = 1$. Now we can finish integrating to get the final answer:

$$\frac{x^2}{2} + 3\ln|x-2| + \ln|x+2| + C.$$

10. (8 points) Solve the following logarithm problems.

(a) Simplify $\log_5(50) + \log_5(\frac{5}{2})$.

> **Solution:**
>
> $$3.$$

(b) Solve the equation $2^{x-1} = e^5$.

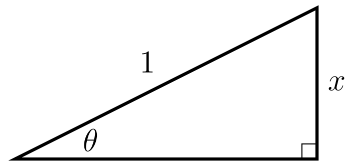
> **Solution:**
>
> $$x = \frac{5}{\ln 2} + 1.$$

(c) Use logarithmic differentiation to find the derivative of $y = (1+x)^x$.

> **Solution:**
>
> $$y' = \left(\frac{x}{1+x} + \ln(1+x)\right)(1+x)^x.$$

11. (8 points) Use the trig substitution $x = \sin\theta$ to evaluate

$$\int x^3\sqrt{1-x^2}\,dx$$



Solution:

$$\frac{(1-x^2)^{5/2}}{5} - \frac{(1-x^2)^{3/2}}{3} + C.$$

12. (4 points) Find each of the infinite sums below using a type of series you know.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^n}$

Solution:

$$e^{-1/2} = \frac{1}{\sqrt{e}}.$$

(b) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \ldots$

Solution:

$$\frac{8}{5}.$$

13. (8 points) I know that the function $f(x) = \dfrac{1}{(x-2)^2}$ is always positive.

Therefore I expected to get a positive answer when I tried to find the area under this curve from $x = 1$ to $x = 3$. However, I got -2 as my answer. Explain what is wrong with my calculation (it may help to draw a picture of $f(x)$).

$$\int_1^3 \frac{1}{(x-2)^2}\,dx = \int_1^3 (x-2)^{-2}\,dx = \left[-(x-2)^{-1}\right]_1^3 = -1 - 1 = -2$$

Solution: There is an asymptote at $x = 2$ so the function is not continuous.

What is the actual value of $\displaystyle\int_1^3 \frac{1}{(x-2)^2}\,dx$? Be sure to show your calculations.

Solution:

$$\lim_{t\to 2^-} \int_1^t \frac{1}{(x-2)^2}\,dx + \lim_{t\to 2^+} \int_t^3 \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{t\to 2^-} \left[\frac{-1}{(x-2)}\right]_1^t + \lim_{t\to 2^+} \left[\frac{-1}{(x-2)}\right]_t^3$$

$$= \left(\infty - \frac{-1}{-1}\right) + \left(\frac{-1}{1} + \infty\right) = \infty.$$

14. (8 points) Find the area between the two curves $f(x) = x^2 - 6x$ and $g(x) = 3 - 4x$.

Solution: Find out where the two functions intersect by setting them equal to each other. Then integrate the top function minus the bottom function. It helps to draw a picture.

(Extra Credit) Estimate the worst case error in using the second degree Taylor polynomial $1 - \frac{x^2}{2}$ to approximate $\cos(0.3)$.