Extra Credit Solution

Claim. If $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle \subset \mathbb{R}^3$ is not a basis, then the vectors $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$ are contained in a 2-dimensional subspace of $\mathbb{R}^3$.

Proof 1 (Direct Proof). If three vectors are not a basis, then they must either not span $\mathbb{R}^3$ or they aren't linearly independent. In fact, if one of these is true, then both are true, since 3 linearly independent vectors in $\mathbb{R}^3$ would span 3-dimensions. So span $\{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\}$ is not $\mathbb{R}^3$. But it is a subspace, so it must be a subspace with dimension less than 3. If it is two dimensional, then we are done. If it is 1 dimensional, that is even better since every one dimensional line is contained in a 2-dimensional plane. $\square$

Proof 2 (Proof by Contradiction). Start by assuming that the conclusion is false, i.e., $\{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\}$ is not contained in a 2-dimensional subspace of $\mathbb{R}^3$. Then the even larger set span $\{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\}$ is not contained in a 2-dimensional subspace either. So it must be a 3-dimensional space (remember, linear spans are vector spaces!). But the only 3-dimensional subspace of $\mathbb{R}^3$ is all of $\mathbb{R}^3$ itself. Not only does this imply that $\{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\}$ spans $\mathbb{R}^3$, but it must also be a basis, since any spanning set for $\mathbb{R}^3$ with three vectors is automatically a basis. That contradicts our original hypothesis: that $\{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\}$ is **not** a basis. This contradiction leads us to reject our assumption at the beginning, so $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$ must be contained in a 2-dimensional subspace. $\square$