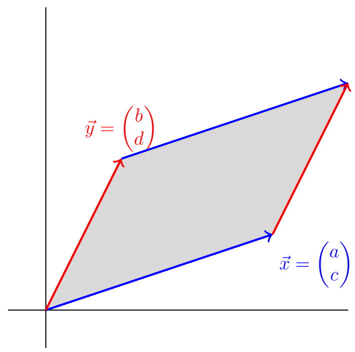
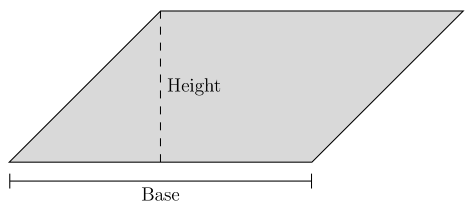
Geometry of Determinants **Math 231 - Workshop**

Theorem 1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*. Then* $|\det(A)|$ *is the area of the parallelogram formed by the vectors* $\vec{x} = \begin{pmatrix} a \\ c \end{pmatrix}$ *and* $\vec{y} = \begin{pmatrix} b \\ d \end{pmatrix}$ *shown below.*



Fact: The area of a parallelogram is the base times the height.



1. Explain why the area of the parallelogram in the theorem is $||\vec{x}||\,||\vec{y}||\sin\theta$ where θ is the angle between the vectors $\vec{x}$ and $\vec{y}$.

2. Use the trig identity $\sin^2\theta + \cos^2\theta = 1$ and the fact that $\cos\theta = \dfrac{\vec{x}\cdot\vec{y}}{||\vec{x}||\,||\vec{y}||}$ to find a formula for $\sin\theta$ in terms of the vectors $\vec{x}$ and $\vec{y}$.

3. Combine your equation for $\sin\theta$ with your equation for the area of the parallelogram to get a formula for the area that doesn't need the angle θ. You should get Area $= \sqrt{||\vec{x}||^2\,||\vec{y}||^2 - (\vec{x}\cdot\vec{y})^2}$.

4. When I plugged a, b, c and d into $\vec{x}$ and $\vec{y}$, I got Area $= \sqrt{a^2d^2 - 2abcd + b^2c^2}$. Why does that prove the theorem?

5. Draw a picture of the parallelogram that you would get if you replaced $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by the matrix $\begin{pmatrix} a & (b+a) \\ c & (d+c) \end{pmatrix}$. Use your picture to explain intuitively why adding a multiple of one column (or row) to another does not change the determinant.

6. Draw a picture of the parallelogram that you would get if you multiplied the first column of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by 2. Use your picture to explain intuitively why doubling one column (or row) doubles the determinant.