Linear Algebra - Math 231 **Midterm 2**

Name: ____________________________

You must show all work to earn full credit. No calculators allowed. If you do not have room in the given space to answer a question, use the back of another page and indicate clearly which work goes with which problem.

Problem	Maximum Points	Your Score
1	16	
2	8	
3	12	
4	8	
5	18	
6	6	
7	8	
8	6	
9	12	
10	6	
Total:	100	

1. (16 points) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 9 & 12 \end{pmatrix}$.

(a) What are the domain and codomain of the linear transformation represented by A?

Solution: Domain $= \mathbb{R}^4$, Codomain $= \mathbb{R}^3$

(b) What is the rank and nullity of A?

Solution: Rank $= 3$, Nullity $= 1$

(c) Find a basis for the rangespace of A.

Solution: Since the rangespace of the transformation defined by A is all of $\mathbb{R}^3$, we can use the standard basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(d) Find a basis for the nullspace of A.

Solution: To find the nullspace, you have to row-reduce A.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 9 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Note that x_1, x_2, x_3 are pivot variables and x_4 is free so the solution of the homogeneous system is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{pmatrix} x_4$$

So the basis for the nullspace is the set with this one direction vector:

$$\left\{ \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{pmatrix} \right\}.$$

2. (8 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that:

$$T(\begin{pmatrix} -1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \qquad T(\begin{pmatrix} 0 \\ 2 \end{pmatrix}) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

(a) What is $T(\begin{pmatrix} 1 \\ 1 \end{pmatrix})$?

Solution:

$$T(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}$$

(b) What is the matrix representation of T with respect to the standard basis?

Solution:

$$\begin{pmatrix} -5 & 1 \\ -4 & 1 \\ -3 & 1 \end{pmatrix}$$

3. (12 points) Determine whether each of the following statements is true or false.

(a) If the rank of a matrix is 3, then the columnspace is $\mathbb{R}^3$.

Solution: False. Just because the rangespace is 3-dimensional doesn't mean it is $\mathbb{R}^3$.

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^4$ has rank 3, then T is onto.

Solution: False. If the rank is 3, then the rangespace can't be all of $\mathbb{R}^4$.

(c) If $T : \mathbb{R}^3 \to \mathbb{R}^4$ has rank 3, then T is one-to-one.

Solution: True. The Rank+Nullity Theorem tells us that the nullity must be 0, so T is one-to-one.

(d) If two vector spaces have different dimensions, then they are linearly independent.

Solution: False. It doesn't even make sense to ask if two vector spaces are linearly independent. That is a property of vectors not vector spaces.

(e) If $T : V \to W$ is one-to-one and onto, then $\dim V = \dim W$.

Solution: True. T is an isomorphism. Isomorphisms preserve dimension.

(f) If $T : V \to W$ is a linear transformation and $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is a set of linearly independent vectors in W, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent subset of V.

Solution: True. If you could find a non-trivial linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0},$$

then

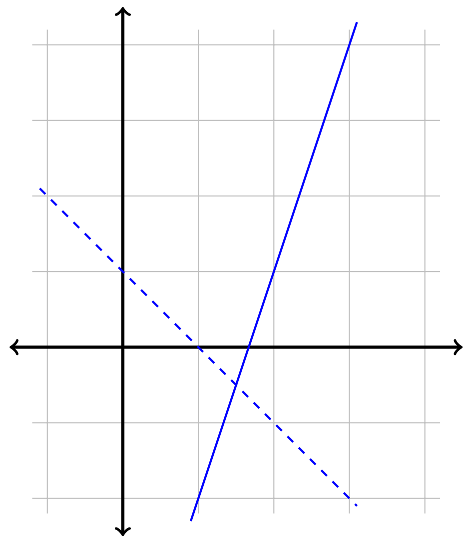
$$c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = T(\vec{0}) = \vec{0}$$

which contradicts the statement that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly independent.

4. (8 points) Let $T(x) = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. What is the image of the line $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix} : t \in \mathbb{R}^2 \right\}$ under the transformation T? Give a formula for the new line and draw a graph showing both the new and the old lines.

Solution:

$$\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix} : t \in \mathbb{R} \right\}$$



5. (18 points) Complete the following definitions.

(a) The **dimension** of a vector space is...

Solution: the number of vectors in a basis for the vector space.

(b) The **nullity** of a linear transformation is...

Solution: the dimension of the nullspace.

(c) A linear transformation is **onto** if...

Solution: the rangespace equals the codomain.

6. (6 points) Explain why the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is not a linear transformation.

Solution: The function f does not respect addition. For example $f(2) = 5$ and $f(3) = 7$ but $f(2 + 3) = 11$ not 12.

7. (8 points) Suppose that $T : \mathbb{R}^6 \to \mathbb{R}^5$ is a linear transformation with nullity 2.

   (a) What is the dimension of the rangespace of T?

   Solution:

   $$4$$

   (b) What is the dimension of the nullspace of T?

   Solution:

   $$2$$

   (c) What is the codomain of T?

   Solution:

   $$\mathbb{R}^5$$

   (d) Is T onto? Explain why or why not.

   Solution: No. The dimension of the rangespace is only 4 but the dimension of the codomain is 5 so they cannot be the same.

8. (6 points) Give an example of a linear transformation that transforms a 2-dimensional plane into a 1-dimensional line. Be sure that you give a complete description of your linear transformation!

Solution: The transformation $T : \mathbb{R}^2 \to \mathbb{R}^1$ with matrix representation

$$T(\vec{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \vec{x}$$

is one example.

9. (12 points) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by

$$T(p(x)) = 2p(x) + \frac{d}{dx}p(x),$$

when $p(x)$ is any polynomial in $\mathcal{P}_2$.

(a) Find $T(x^2), T(x)$, and $T(1)$.

Solution:

$$T(x^2) = 2x^2 + 2x \qquad T(x) = 2x + 1 \qquad T(1) = 2$$

(b) Find the matrix representation of T with respect to the basis $B = \langle x^2, x, 1 \rangle$.

Solution:

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

(c) What are the rank and nullity of T?

Solution: Rank = 3, Nullity = 0

10. (6 points) Find the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates every vector in $\mathbb{R}^2$ by 90° counterclockwise.

Solution:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$