Linear Algebra **Midterm 2 Review Problems**

1. This problem is about the following system of equations:

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 + 1x_4 &= a \\ -6x_1 + 4x_2 + 4x_3 + 5x_4 &= b \\ -15x_1 + 10x_2 + 6x_3 + 9x_4 &= c \\ 18x_1 - 12x_2 - 4x_3 - 8x_4 &= d \\ -9x_1 + 6x_2 + 2x_3 + 4x_4 &= e \end{aligned}$$

After some elementary row operations, the augmented matrix for this system becomes:

$$\left(\begin{array}{cccc|c} 1 & -\frac{2}{3} & 0 & -\frac{1}{4} & \frac{1}{12}c - \frac{1}{4}e \\ 0 & 0 & 1 & \frac{7}{8} & \frac{3}{8}c - \frac{5}{8}e \\ 0 & 0 & 0 & 0 & a - c + 2e \\ 0 & 0 & 0 & 0 & b - c + e \\ 0 & 0 & 0 & 0 & d + 2e \end{array}\right)$$

The following questions refer to the *coefficient matrix* of the system.

(a) Give the dimension and a basis for the null space of the coefficient matrix.

(b) Give the dimension and a basis for the row space of the coefficient matrix.

(c) Give the dimension and a basis for the column space of the coefficient matrix.

2. Make sure that you can define each of the following mathematical concepts.

(a) Dimension

(b) Linear Transformation

(c) Nullity

(d) Nullspace

(e) Rangespace

(f) One-to-one

(g) Onto

3. Suppose that S is a subspace of $\mathbb{R}^{100}$. Suppose S contains a linearly independent subset with 40 vectors and S also contains a spanning set with 60 vectors. Given this information, what can you tell me about the dimension of S? Explain.

4. Let $T : V \to W$ be a linear transformation. Prove that if $v_1, v_2 \in V$, then the midpoint of the line segment connecting v_1 to v_2 gets mapped to the midpoint of the line segment connecting $T(v_1)$ to $T(v_2)$ by T.

5. Let $B = \langle x^3, x^2, x^1, 1\rangle$ be a basis for the vector space $\mathcal{P}_3$.

(a) What is $\text{Rep}_B(5x^3 - 4x^2 + 3x - 2)$?

(b) What is $T(x^3 - x^2 + x)$ if $T : \mathcal{P}_3 \to \mathcal{P}_3$ has the matrix representation

$$\text{Rep}_{B,B}(T) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}?$$

6. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with matrix representation

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 3 & 6 & 0 & -2 \\ -5 & -10 & 0 & 5 \end{pmatrix}$$

(a) What is the rank of T?

(b) Is the vector $\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$ in the rangespace of T?

(c) What is the nullity of T?

(d) What is the nullspace of T?

7. Find a matrix representation for the linear transformation that reflects every vector in $\mathbb{R}^2$ across the line $y = -x$. Assume that the first coordinate of vectors in $\mathbb{R}^2$ is the x-coordinate, and the second coordinate is the y-coordinate.

8. What are the possible shapes for the image of a 2-dimension plane under a linear transformation?

9. Explain how you can tell that the rank of the matrix below must be 2 without bothering to row-reduce the matrix to echelon form.

$$\begin{pmatrix} 5 & 2 & -8 & -1 & 0 \\ 7 & -2 & 6 & 0 & -2 \end{pmatrix}$$

10. Find the image of the line $\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} t : t \in \mathbb{R} \right\}$ under the linear transformation

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and draw a picture of both the original and transformed lines.