Linear Algebra **Midterm 3 Review Problems**

1. Make sure that you can define each of the following mathematical concepts.

   (a) Elementary reduction matrix

   Solution: A matrix that can be obtained from the identity matrix by one Gaussian elimination step.

   (b) Invertible matrix

   Solution: A matrix A is invertible if there is a matrix B such that $AB = BA = I$.

   (c) Inverse of a matrix

   Solution: The inverse A is a matrix B such that $AB = BA = I$.

   (d) Similar matrices

   Solution: We say that A is similar to B if there is an invertible matrix S such that $A = S^{-1}BS$.

   (e) Characteristic polynomial

   Solution: The characteristic polynomial is $\det(A - \lambda I)$.

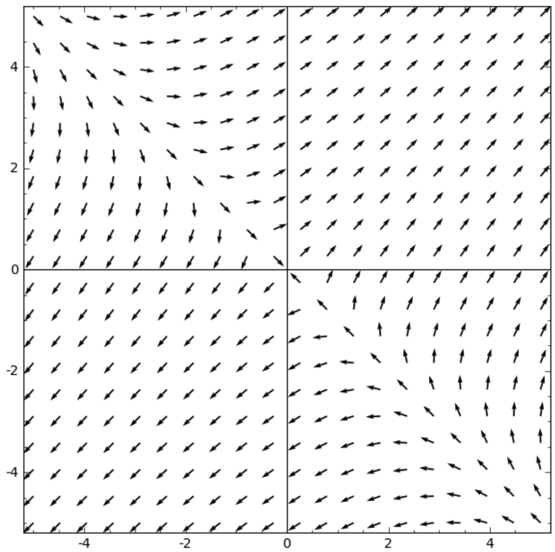
   (f) Eigenvector

   Solution: An eigenvector of A is any **nonzero** vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ.

   (g) Eigenvalue

   Solution: An eigenvalue of A is any scalar λ such that $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.

2. Which matrix below corresponds to the following direction field?



A. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

B. $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

C. $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

3. Find the determinants of the following matrices.

(a) $\begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 5 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$

4. Prove that if λ is an eigenvalue of a matrix A, then $c\lambda$ is an eigenvalue of the matrix cA for any scalar c.

5. For any matrix of the form $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ where $a, b, c \in \mathbb{R}$, show that the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector. What is its corresponding eigenvalue?

6. Find the eigenvalues and eigenvectors of the following matrices.

   (a) $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2i & -1 \\ 0 & -i \end{pmatrix}$

7. Let $a, b \in \mathbb{R}$. Show that the 2-by-2 matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ has eigenvalues $a \pm bi$ by showing that $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ are eigenvectors of A.

8. For each elementary reduction matrix, describe in words the row operation that the matrix represents.

   (a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

9. The matrix $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ has only one eigenvalue, 2. What is the dimension of the eigenspace corresponding to 2? What is a basis for that eigenspace?

10. Suppose that A is a 4-by-4 matrix and $\vec{v}$ is an eigenvector of A with corresponding eigenvalue 3.

    (a) Is $-2\vec{v}$ an eigenvector of A? If so, what is its corresponding eigenvalue?

    (b) Is $\vec{v}$ an eigenvector of $5A$. If so, what is its corresponding eigenvalue?

    (c) Is $\vec{v}$ and eigenvector of A^2. If so, what is its corresponding eigenvalue?

11. A is a 3-by-3 matrix with eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ and corresponding eigenvalue $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Find the matrix A.

12. The unit circle (i.e., the circle with radius 1 centered at the origin in $\mathbb{R}^2$) has area π. What is the area of the image of the unit circle after being transformed by the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$?

13. Suppose that $A = B \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} B^{-1}$, where $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

    (a) Find A

    (b) Find A^6.