Newton's Method **Math 342**

Last time, we proved that if $f \in C^2[a,b]$ has a root $r \in [a,b]$ and there are constants $L, M > 0$ such that $|f'(x)| \geq L$ and $|f''(x)| \leq M$ for all $x \in [a,b]$, then

$$|x_{n+1} - r| \leq \frac{M}{2L}|x_n - r|^2$$

whenever $x_n \in [a,b]$. From this theorem, we also showed that

$$|x_n - r| \leq \left(\frac{M}{2L}\right)^{2^n - 1} |x_0 - r|^{2^n}$$

for all n as long as the Newton's method iterates x_n stay inside the interval $[a,b]$.

1. Suppose we have a function $f \in C^2(\mathbb{R})$ such that $|f'(x)| \geq 2$ and $|f''(x)| \leq 5$ for all x.
   (a) How bad could the error $|x_n - r|$ get after $n = 10$ iterations of Newton's method if we start with an initial guess x_0 such that $|x_0 - r| = 1$?

   (b) How bad could the error $|x_n - r|$ get after $n = 10$ iterations of Newton's method if $|x_0 - r| = 0.5$?

2. Write a Python function to apply Newton's method to any function f. Your function should input four parameters: `f` the function, `Df` its derivative, `x0` an initial guess for the root, and `n` the number of times you want to iterate. It should then output the value of the n^{th} iterate of Newton's method.

3. Use your function to find all 3 roots of the polynomial $x^3 - 5x + 3$. Your roots should each be accurate to at least 10 decimal places.

4. Notice that one of the roots of the polynomial $x^3 - 5x + 3$ is in the interval $[-3, -2]$. On this interval, what is the minimum value of $|f'(x)|$? What is the maximum value of $|f''(x)|$?

5. Use the minimum value for $|f'(x)|$ as L and the maximum value of $|f''(x)|$ as M to estimate how close your first guess x_0 should be to the root $r \in [-3, -2]$ in order for Newton's method the safely converge.