Numerical Analysis - Math 342 **Midterm 2 Review**

The following problems are similar to ones you might see on the midterm exam.

1. Use the method of divided differences to find the Newton basis interpolating polynomial for the points $(0,0)$, $(1,1)$, and $(4,2)$.

2. What is the interpolating polynomial above written in terms of the Lagrange basis polynomials?

3. Write down the Vandermonde matrix system for these same points $(0,0)$, $(1,1)$, $(4,2)$ to find the coefficients of the interpolating polynomial in the standard basis. You don't need to solve the Vandermonde matrix system.

4. The second degree interpolating polynomial for the function $f(x) = 1/x$ on the interval $[1, 3]$ with three equally spaced nodes ($x_0 = 1,\ x_1 = 2,\ x_2 = 3$) is

$$p_2(x) = \tfrac{1}{6}x^2 - x + \tfrac{11}{6}.$$

The error formula for interpolating polynomials with equally spaced nodes is

$$|f(x) - p_n(x)| \leq \frac{h^{n+1}}{4(n+1)} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$$

where $h = \frac{b-a}{n}$. Use this formula to find an upper bound on the error in using the polynomial p_2 to approximate $f(x) = 1/x$ on the interval $[1, 3]$.

5. The formula for Simpson's method (not composite) on an interval is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a) + 4f(m) + f(b)\right) - \frac{h^5}{90} f^{(4)}(\xi)$$

where $h = \frac{b-a}{2}$ and $m = \frac{a+b}{2}$ and ξ is some value between a and b. Use this rule to approximate area under the function $y = e^x$ on the interval $[0, 2]$ and estimate the error in the approximation.

6. To estimate the area under a curve using Gaussian quadrature you need to convert the function to an equivalent integral on the interval $[-1, 1]$. Then you can use Gaussian quadrature with any number of nodes. The formula for Gaussian quadrature with $n = 3$ nodes is

$$\int_{-1}^{1} f(x)\,dx \approx \tfrac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \tfrac{8}{9} f(0) + \tfrac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right).$$

Find an integral on $[-1, 1]$ that is equal to $\int_0^2 e^x\,dx$ and then use Gaussian quadrature to estimate the value of the integral.

7. To approximate the improper integral $\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$, we could use $\displaystyle\int_1^N \frac{\sin^2(x)}{x^2}\,dx$ for some large N. Find a value of N that is large enough so that the absolute error in the approximation is less than 10^{-12}. Make sure to show why your N works.

8. The normal equation to find the coefficients of a (discrete) least square regression model is

$$X^T X b = X^T y.$$

Suppose you want the best fit linear function $\hat{y} = b_0 + b_1 x$ to approximate the points $(-2, 3)$, $(0, 2)$, $(2, 0)$.

(a) What is the matrix X and the vector y in the normal equation above?

(b) Compute $X^T X$ and $X^T y$.

(c) Solve the normal equations to find the slope and y-intercept of the regression line $\hat{y} = b_0 + b_1 x$.

9. The Legendre polynomials are a family of orthogonal functions on the interval $[-1, 1]$. The first three Legendre polynomials are

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1).$$

Using the Legendre functions as a basis, find the best fit (continuous least squares) 2nd degree polynomial approximation of the function $\cos x$ on the interval $[-1, 1]$. You can use the following integrals to help

$$\int_{-1}^{1} P_0(x) \cos x \, dx = 1.683 \qquad \int_{-1}^{1} P_1(x) \cos x \, dx = 0$$

and

$$\int_{-1}^{1} P_2(x) \cos x \, dx = -0.124 \qquad \int_{-1}^{1} P_2(x)^2 \, dx = 0.4.$$