Math 444 - Homework 4 **Name:** ______________________

Let $z = x + iy$ where $x, y \in \mathbb{R}$. For each of the following functions, use the Cauchy-Riemann equations to determine the set of all $z \in \mathbb{C}$ where the function is differentiable. At the points where the function is differentiable, what is the derivative?

1. $f(z) = z^2 - (\overline{z})^2$.

2. $f(z) = x^2 + iy^2$.

3. $f(z) = e^{-x}e^{-iy}$.

4. $f(z) = z \operatorname{Im} z$.

5. Find the derivative of the function $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ are constants and $ad - bc \neq 0$. When is $T'(z) = 0$? Hint: Use the quotient rule.

6. Suppose that $f(z) = e^z$. Let $\gamma_1(t) = t + i/t$ and $\gamma_2(t) = t + ti$, where $t > 0$. Consider the paths $f(\gamma_1(t))$ and $f(\gamma_2(t))$. Use the Chain Rule formula:

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t)) \cdot \gamma'(t)$$

to find the tangent vectors $\frac{d}{dt} f(\gamma_1(t))$ and $\frac{d}{dt} f(\gamma_2(t))$ when $t = 1$. Are the tangent vectors orthogonal? Why or why not?

7. If f is holomorphic in an open connected set $G \subseteq \mathbb{C}$ and f is always real-valued, then prove that $f'(z) = 0$ everywhere on G. Hint: Use the Cauchy-Riemann equations.

8. If $f(z)$ and $\overline{f(z)}$ are both holomorphic on an open connected set $G \subseteq \mathbb{C}$, show that $f(z)$ is constant in G.