Math 444 - Sequences and Convergence **Workshop**

- A **sequence** is a function $s : \mathbb{N} \to \mathbb{C}$. Notation: we write s_n instead of $s(n)$ for sequences.
- A sequence s_n **converges** to $z \in \mathbb{C}$ if for every $\epsilon > 0$, there is an N large enough so that $|s_n - z| < \epsilon$ for every $n \geq N$. In other words, for every open disk centered at z, there are only finitely many n with s_n outside the disk.
- A sequence s_n is **bounded** if there is a constant $M > 0$ such that $|s_n| \leq M$ for every $n \in \mathbb{N}$.

Notice that a sequence s_n converges to z if and only if $s_n - z$ converges to 0.

1. Every converging sequence is bounded. But the converse is not true, that is, there are sequences which are bounded, but do not converge. Find an example.

2. Show that the sequence $s_n = \frac{1}{\sqrt{n}} e^{i\frac{\pi\sqrt{n}}{6}}$ converges to 0. For any fixed $\epsilon > 0$ find a formula for an N that is big enough so that $|s_n - 0| < \epsilon$ when $n \geq N$. Your formula should be a function of ϵ.

3. Suppose a_n and b_n are two sequences that both converge to zero. Prove that $a_n + b_n$ converges to 0.

4. If a_n converges to zero and b_n is bounded, prove that the sequence $a_n \cdot b_n$ converges to zero.

5. Suppose a_n converges to a and b_n converges to b. Prove that:

   (a) $a_n + b_n$ converges to $a + b$.

   (b) $a_n \cdot b_n$ converges to $a \cdot b$.

Let A be a subset of $\mathbb{C}$. A function $f : A \to \mathbb{C}$ is **continuous at** $a \in A$ if for every sequence $s_n \in A$ that converges to a, the sequence $f(s_n)$ converges to $f(a)$. We say that f is **continuous on** A (or just **continuous**) if f is continuous at every $a \in A$.

6. Suppose that $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ are both continuous at $a \in A$. Prove that:

   (a) $f + g$ is continuous at a.

   (b) $f \cdot g$ is continuous at a.

7. Suppose that $g : A \to \mathbb{C}$ is continuous on A and $f : g(A) \to \mathbb{C}$ is continuous on $g(A)$. Prove that $f(g(z))$ is continuous on A.