Points and Vectors

Lecture 15

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Outline

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3D Homogeneous Coordinates

- The 3-dimensional point (x, y, z) may be written in 4D homogeneous coordinates as (X, Y, Z, W), where

$$x = X/W,$$
$$y = Y/W,$$
$$z = Z/W.$$

- Thus, the points $(1, 2, 3, 1)$, $(2, 4, 6, 2)$, and $(-5, -10, -15, -5)$ all represent the same 3D point $(1, 2, 3)$.

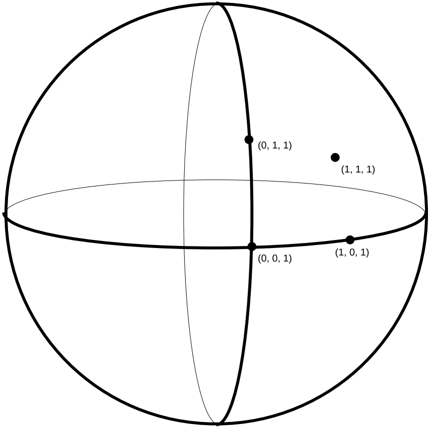
Homogeneous Coordinates

- Homogeneous coordinates are used in projective geometry to carry out projections.
- They are used in compute graphics for the same reason.
- At one stage in the processing of a vertex, x, y, and z are divided by w.
- This is called the homogeneous divide and it occurs late in the processing, when the 3D scene is projected onto a 2D plane.

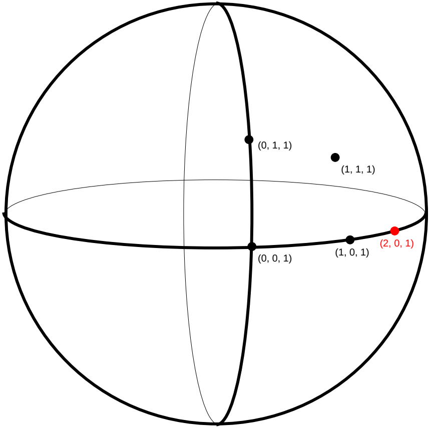
Outline

The Half-Sphere Model

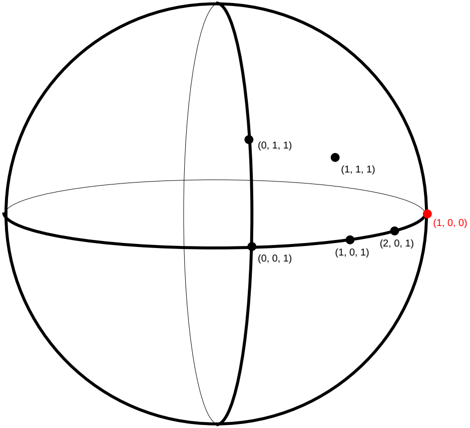
- The half-sphere is a good model of the projective plane.
- Polar-opposite points on the sphere are considered to be the same point.
- Thus, only half of the sphere is needed for the model (or we work with equivalence classes of antipodal points).
- "Lines" in the model are great circles (equators) on the sphere.



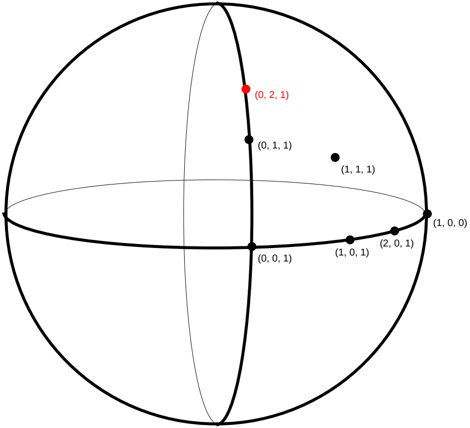
The affine points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$



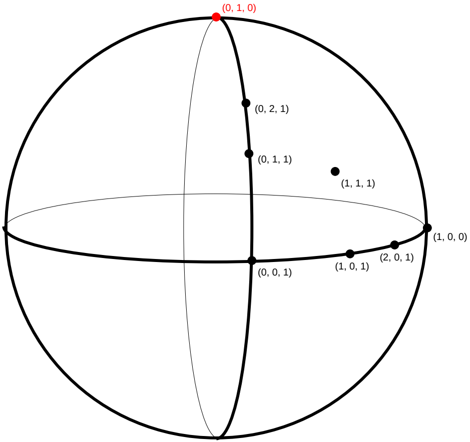
The affine point $(2, 0)$



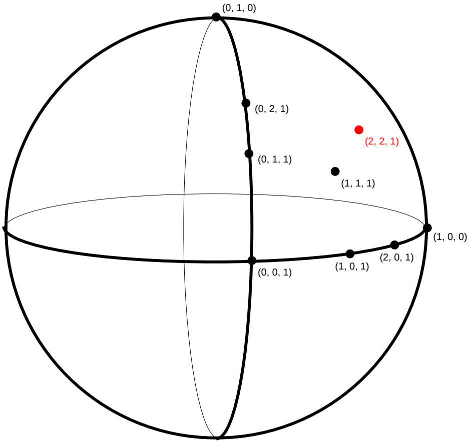
The point at infinity $(1, 0, 0)$



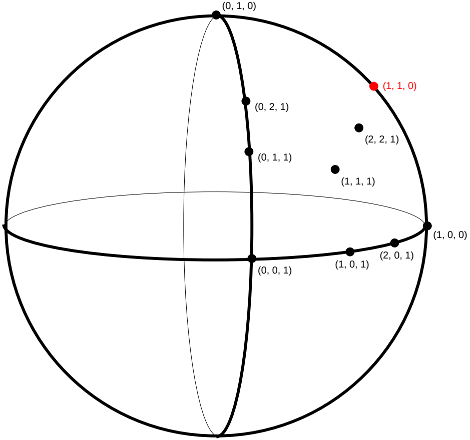
The affine point $(0, 2)$



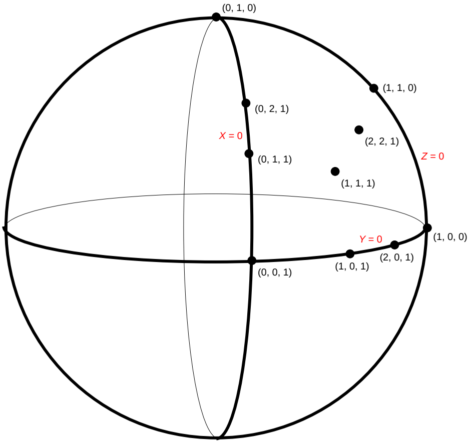
The point at infinity $(0, 1, 0)$



The affine point $(2, 2)$



The point at infinity $(1, 1, 0)$



The X-, Y-, and Z-axes

The Line at Infinity

- The Z-axis is called the line at infinity.
- Parallel lines in the affine plane are great circles that meet on the Z-axis in the half-plane model.
- In that sense, parallel lines *literally* meet at infinity.

The Line at Infinity

- For example, consider the parallel lines $y = 1$ and $y = 2$.
- In homogeneous coordinates, the equations are $Y = Z$ and $Y = 2Z$.
- The solution is $Y = Z = 0$ and X can have any value (so it might as well be 1).
- Thus, the point of intersection is $(1, 0, 0)$, which is the point at infinity on the X-axis.

Other Interesting Examples

- Where do the two branches of the parabola $y = x^2$ meet the line at infinity?
- The equation $\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$ represents a circle of radius $\frac{1}{2}$ with center at $\left(\frac{1}{2}, 0\right)$. Make the y-axis the line at infinity and find the equation of this circle.
- For the same circle, make the x-axis the line at infinity and find the equation of this parabola.

Outline

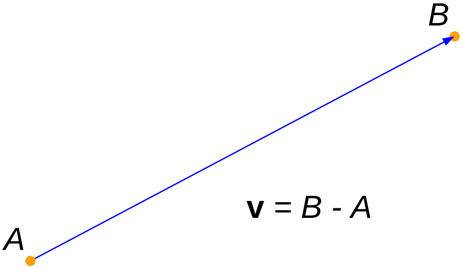
- Points and vectors both may be created as **`vec3`** objects.
- However, as **`vec4`** objects, which they eventually will be,
  - For points, $w \neq 0$ (usually $w = 1$).
  - For vectors, $w = 0$.
- In a projective sense, a vector is a "point at infinity."

Point and Vector Arithmetic

- Let P and Q be points, $\mathbf{u}$ and $\mathbf{v}$ be vectors, and c be a scalar.
  - $\mathbf{u} + \mathbf{v}$ is a vector.
  - $\mathbf{u} - \mathbf{v}$ is a vector.
  - $P - Q$ is a vector.
  - $P + \mathbf{v}$ and $\mathbf{v} + P$ are a points.
  - $P - \mathbf{v}$ is a point.
  - $c\mathbf{v}$ is a vector.

Point subtraction

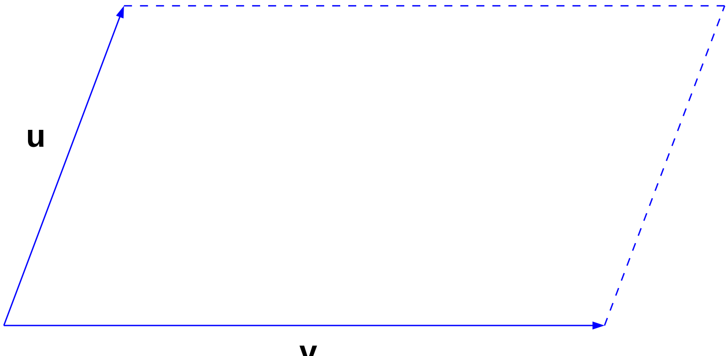
Point and Vector Arithmetic



$\mathbf{v} = B - A$

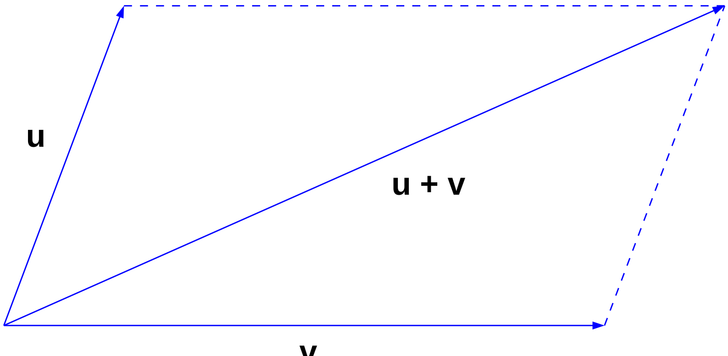
Point subtraction

Point and Vector Arithmetic



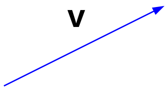
Vector addition

Point and Vector Arithmetic



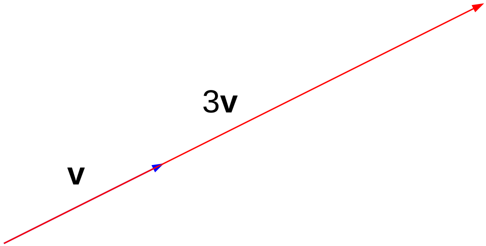
Vector addition

Point and Vector Arithmetic



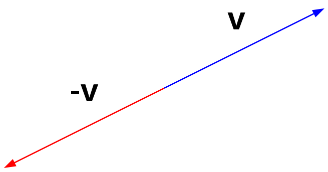
Scalar multiplication

Point and Vector Arithmetic



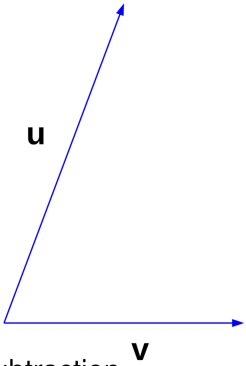
Scalar multiplication

Point and Vector Arithmetic

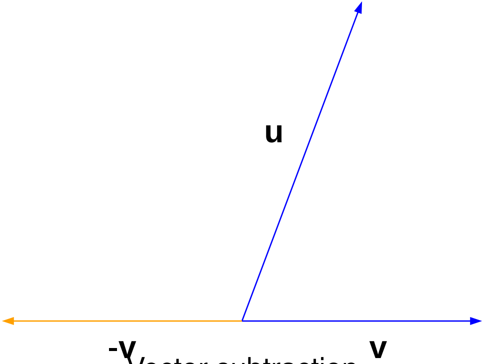


Scalar multiplication

Point and Vector Arithmetic

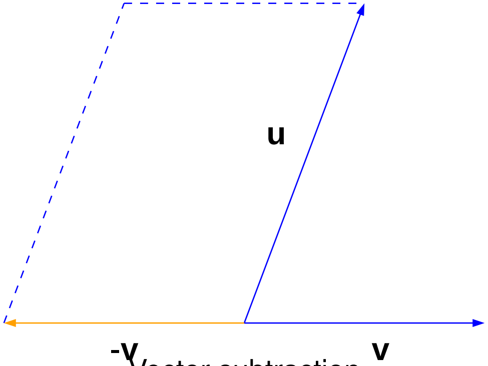


Vector subtraction



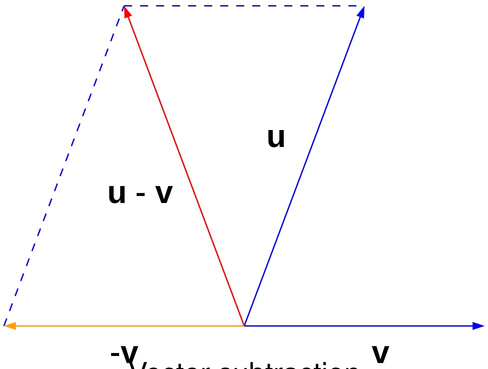
Vector subtraction

Point and Vector Arithmetic



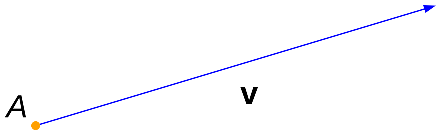
Vector subtraction

Point and Vector Arithmetic



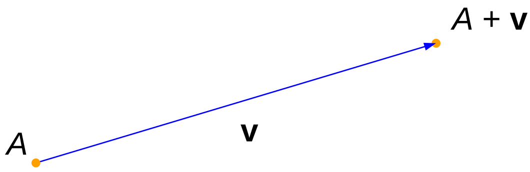
Vector subtraction

Point and Vector Arithmetic



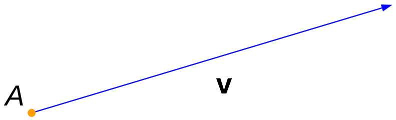
Point-vector addition

Point and Vector Arithmetic



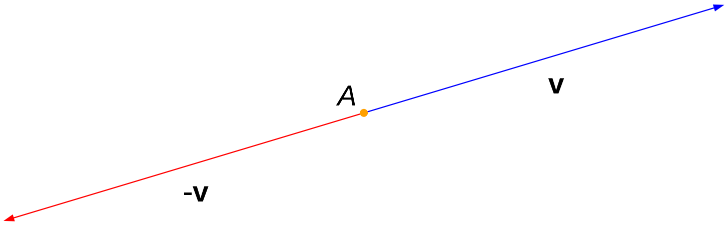
Point-vector addition

Point and Vector Arithmetic



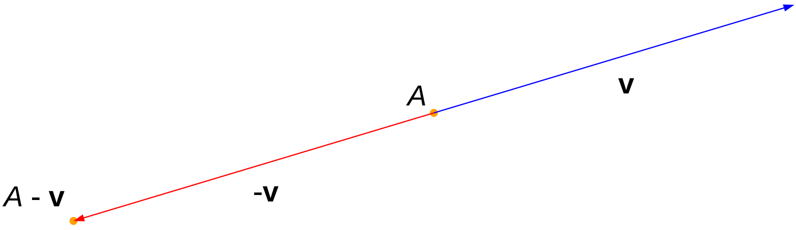
Point-vector subtraction

Point and Vector Arithmetic



Point-vector subtraction

Point and Vector Arithmetic



Point-vector subtraction

Point and Vector Arithmetic

- What about...
  - $\mathbf{v} - P$?
  - $P + Q$?
  - cP?

Point and Vector Arithmetic

- What about...
  - $\mathbf{v} - P$?
  - $P + Q$?
  - cP?
- Hint: Consider the homogeneous coordinate.

Point and Vector Arithmetic

- Let P, Q, and R be points and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors.
- Which of the following statements are true?
  - $P - (Q - R) = (P - Q) + R = R - (Q - P)$
  - $P - (Q - \mathbf{v}) = (P - Q) + \mathbf{v}$
  - $P - (Q + \mathbf{v}) = (P - Q) - \mathbf{v}$
  - $P + (Q - \mathbf{v}) = (P + Q) - \mathbf{v}$
  - $P + (\mathbf{u} + \mathbf{v}) = (P + \mathbf{u}) + \mathbf{v}$
  - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

Outline

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Definition (The Dot Product)

The magnitude of a vector is its length. It is given by the distance formula.

- Let $\mathbf{v} = (v_1, v_2, v_3)$.
- The magnitude of $\mathbf{v}$, denoted $|\mathbf{v}|$, is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Normalized Vectors

- To normalize a vector, we divide it by its length.
- That is, for any vector $\mathbf{v} \neq \mathbf{0}$, the unit vector $\mathbf{n}$ with the same direction as $\mathbf{v}$ is

$$\mathbf{n} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Outline

The Dot Product

Definition (The Dot Product)

The dot product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined to be

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

- Note that the dot product of two vectors is a scalar.

Algebraic Properties of the Dot Product

- Let **u**, **v**, and **w** be vectors and let c be a real number and let θ be the angle between **u** and **v**.
- Then

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \\
(c\mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v}) \\
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \\
\mathbf{v} \cdot \mathbf{v} &= |\mathbf{v}|^2 \\
\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}||\mathbf{v}| \cos \theta
\end{aligned}$$

Dot Products and Angles

- A consequence of the last property is that
  - $\mathbf{u} \cdot \mathbf{v} > 0$ if and only if $0^\circ \leq \theta < 90^\circ$ (acute angle).
  - $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\theta = 90^\circ$ (right angle).
  - $\mathbf{u} \cdot \mathbf{v} < 0$ if and only if $90^\circ < \theta \leq 180^\circ$ (obtuse angle).
- This is of *enormous* importance in computer graphics.

Orthogonal Projections

Definition (Orthogonal Projection)

The orthogonal projection of a vector $\mathbf{u}$ onto a vector $\mathbf{v}$ is the vector

$$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}.$$

- For example, the projection of $\mathbf{u} = (5, 0, 2)$ onto $\mathbf{v} = (3, 4, 5)$ is

$$\begin{aligned}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} &= \left(\frac{5\cdot 3 + 0\cdot 4 + 2\cdot 5}{3\cdot 3 + 4\cdot 4 + 5\cdot 5}\right)(3, 4, 5)\\ &= \left(\frac{25}{50}\right)(3, 4, 5)\\ &= \left(\frac{3}{2}, \frac{4}{2}, \frac{5}{2}\right).\end{aligned}$$

Outline

The Cross Product

Definition (Cross Product)

The cross product of vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined to be the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

- To find normal vectors, we need the cross product.
- Note that the cross product of vectors is a vector, not a scalar.

The Cross Product

u_1	u_2	u_3
v_1	v_2	v_3

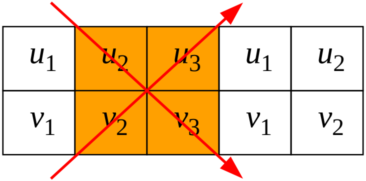
An easy way to remember the cross product.

The Cross Product

u_1	u_2	u_3	u_1	u_2
v_1	v_2	v_3	v_1	v_2

Duplicate the first and second columns.

The Cross Product

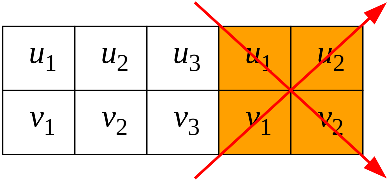


Find this 2×2 determinant for the first component.

The Cross Product

u_1	u_2	u_3	u_1	u_2
v_1	v_2	v_3	v_1	v_2

Find the next 2×2 determinant for the second component.



Find the last 2×2 determinant for the third component.

Algebraic Properties of the Cross Product

- Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c be a real number and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$
$$(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v})$$
$$\mathbf{v} \times \mathbf{v} = \mathbf{0}$$
$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$$
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

The Right-hand Rule

- The right-hand rule helps us remember which way $\mathbf{u} \times \mathbf{v}$ points.
- Arrange the thumb, index finger, and middle finger so that they are mutually orthogonal.
- Let the thumb represent $\mathbf{u}$ and the index finger represent $\mathbf{v}$.
- Then the middle finger represents $\mathbf{u} \times \mathbf{v}$.

Finding Surface Normals

Example (Finding Surface Normals)

Given a triangle ABC, where $A = (1, 1, 2)$, $B = (3, 1, 5)$, and $C = (1, 0, 4)$, find a unit vector $\mathbf{N}$ that is normal to the surface.

Example

Example (Finding Surface Normals)

- Let

$$\mathbf{u} = B - A = (2, 0, 3)$$
$$\mathbf{v} = C - A = (0, -1, 2)$$

- Then $\mathbf{n} = \mathbf{u} \times \mathbf{v} = (3, -4, -2)$.
- $|\mathbf{n}| = \sqrt{29}$, so the unit normal is

$$\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|} = \left(\frac{3}{\sqrt{29}}, -\frac{4}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right).$$

Outline

The vec3 Class

Vector Functions

```
float length(vecn v);
float dot(vecn u, vecn v);
vec3 cross(vec3 u, vec3 v);
```

- In the **vec** classes (**vec2**, **vec3**, **vec4**), there are member functions for the length and the dot product.
- The cross product applies to **vec3** objects only.

Outline

Assignment

Assignment

- Read pp. 207 - 210, Homogeneous Coordinates.