Digital Logic

Lecture 18
Section C.2 (on CD)

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Outline

Digital Logic

- The digital system hierarchy
  - Transistors.
  - Gates and flip-flops.
  - Functional units (registers, memories, arithmetic units, etc.).
  - Systems (e.g., computers).

Digital Logic

- Logic systems are combinational or sequential.
- Combinational systems have no internal memory.
  - The same inputs always produce the same outputs.
- Sequential systems have internal memory.
  - The same inputs may produce different outputs, depending on the values stored.

Gates

- A gate is a simple electronic system that implements a logical operation.
- The direction of information flow is from the input terminals to the output terminal.
- The number of input and output terminals is finite and they carry binary-valued signals.
- The transformation of input signals to output signals can be modeled by a boolean expression.

Gates

- Basic combinatorial circuits
  - AND gate
  - OR gate
  - Inverter (NOT gate)
  - XOR gate
  - Multiplexor
  - Decoder

Gates

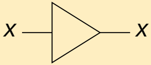
- Combinatorial circuits can be represented in different ways.
  - Truth tables
  - Boolean expressions
  - Logic diagrams
- We can transform any one of these in any other.

Buffer

Buffer

x

x	x
0	0
1	1



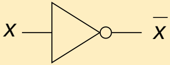
Boolean expression   Truth table   Logic diagram

NOT Gates

NOT Gates

$\overline{x}$

x	$\overline{x}$
0	1
1	0



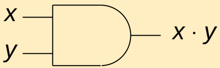
Boolean expression   Truth table   Logic diagram

AND Gates

AND Gates

$x \cdot y$

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1



Boolean expression   Truth table   Logic diagram

NAND Gates

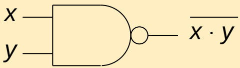
NAND Gates

$\overline{x \cdot y}$

Boolean expression

x	y	$\overline{x \cdot y}$
0	0	1
0	1	1
1	0	1
1	1	0

Truth table



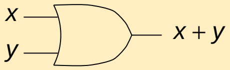
Logic diagram

OR Gates

OR Gates

$x + y$

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1



Boolean expression

Truth table

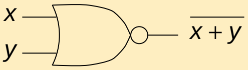
Logic diagram

NOR Gates

NOR Gates

$\overline{x+y}$

x	y	$\overline{x+y}$
0	0	1
0	1	0
1	0	0
1	1	0



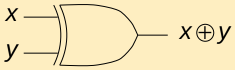
Boolean expression Truth table Logic diagram

XOR Gates

XOR Gates

$x \oplus y$

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0



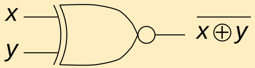
Boolean expression Truth table Logic diagram

NXOR Gates

NXOR Gates

$\overline{x \oplus y}$

x	y	$\overline{x \oplus y}$
0	0	1
0	1	0
1	0	0
1	1	1



Boolean expression Truth table Logic diagram

Outline

- In boolean algebra
  - There are two boolean values: 1 and 0 (or true and false).
  - Variables take on only those values.
  - There are two binary operators: $+$ and $\cdot$, representing “and” and “or.”
  - There is one unary operator: $\bar{\ }$, representing “not.”
  - There are 19 axioms, defining the behavior of the operators.
- The axioms exhibit duality: Each axiom has its dual, by reversing $+$ and $\cdot$ and reversing 1 and 0.

Boolean Algebra Basic Properties

- Commutativity

Boolean Algebra	C++
$a + b = b + a$	`a + b == b + a`
$a \cdot b = b \cdot a$	`a*b == b*a`

- Associativity

Boolean Algebra	C++
$(a + b) + c = a + (b + c)$	`(a + b) + c == a + (b + c)`
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	`(a*b)*c == a*(b*c)`

- Distributivity

Boolean Algebra	C++
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	`a*(b + c) == a*b + a*c`
$a + (b \cdot c) = (a + b) \cdot (a + c)$	Oops!

Boolean Algebra Basic Properties

- Idempotence

Boolean Algebra	C++
$a + a = a$	`a || a == a`
$a \cdot a = a$	`a && a == a`

- Identity

Boolean Algebra	C++
$a + 0 = a$	`a || false == a`
$a \cdot 1 = a$	`a && true == a`

- Annihilation

Boolean Algebra	C++
$a + 1 = 1$	`a || true == true`
$a \cdot 0 = 0$	`a && false == false`

Boolean Algebra Basic Properties

- Complementation

Boolean Algebra	C++
$a + \overline{a} = 1$	`a || !a == true`
$a \cdot \overline{a} = 0$	`a && !a == false`

- DeMorgan's Laws

Boolean Algebra	C++
$\overline{a + b} = \overline{a} \cdot \overline{b}$	`!(a || b) == !a && !b`
$\overline{a \cdot b} = \overline{a} + \overline{b}$	`!(a && b) == !a || !b`

- Absorption

Boolean Algebra	C++
$a \cdot (a + b) = a$	`a && (a || b) == a`
$a + (a \cdot b) = a$	`a || (a && b) == a`

- Double negation (self-dual)

Boolean Algebra	C++
$\overline{\overline{a}} = a$	`!!a == a`

Outline

Truth Tables

Truth Tables

Input Variables	Output Variables
⋮	⋮

- Create one column for each input variable and one column for each output variable (or expression).
- In the input columns, list every possible combination of values.
- In the output columns, write the computed values.

Example (Two Inputs, Two Outputs)

Input		Intermediate		Output	
x	y	$x \cdot y$	$x + y$	$x \cdot y + y$	$(x + y) \cdot y$
0	0	0	0	0	0
0	1	0	1	1	1
1	0	0	1	0	0
1	1	1	1	1	1

- Note that this example demonstrates that

$$x \cdot y + y = (x + y) \cdot y$$

Truth Tables

Example (Three Inputs, One Output)

Input			Intermediate	Output
x	y	z	$x+y$	$(x+y)\cdot z$
0	0	0	0	0
0	1	0	1	0
1	0	0	1	0
1	1	0	1	0
0	0	1	0	0
0	1	1	1	1
1	0	1	1	1
1	1	1	1	1

Outline

Combinational Circuits

Example (Combinational Circuit)

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

- Using AND, OR, and NOT gates, write a boolean expression for this function.

Combinational Circuits

- We may begin by writing the expression in disjunctive normal form:
  - A disjunction of clauses.
  - Each clause is a conjunction of literals, exactly one literal for each input.
  - Each literal is either an input variable or is inverse (x or $\overline{x}$).
- The disjunctive normal form in this example is

$$\overline{A}BC + A\overline{B}\overline{C} + A\overline{B}C + AB\overline{C} + ABC.$$

- Then simplify the expression using the rules of Boolean algebra.

Combinational Circuits

Example (Combinational Circuit)

$$\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC$$

Combinational Circuits

Example (Combinational Circuit)

$$\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC$$
$$= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C)$$

Combinational Circuits

Example (Combinational Circuit)

$$\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1
\end{aligned}$$

Combinational Circuits

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB
\end{aligned}
$$

Combinational Circuits

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC & \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB \\
&= \overline{A}BC + A(\overline{B} + B)
\end{aligned}
$$

Combinational Circuits

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB \\
&= \overline{A}BC + A(\overline{B} + B) \\
&= \overline{A}BC + A \cdot 1
\end{aligned}
$$

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB \\
&= \overline{A}BC + A(\overline{B} + B) \\
&= \overline{A}BC + A \cdot 1 \\
&= \overline{A}BC + A
\end{aligned}
$$

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB \\
&= \overline{A}BC + A(\overline{B} + B) \\
&= \overline{A}BC + A \cdot 1 \\
&= \overline{A}BC + A \\
&= (\overline{A} + A)(BC + A)
\end{aligned}
$$

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB \\
&= \overline{A}BC + A(\overline{B} + B) \\
&= \overline{A}BC + A \cdot 1 \\
&= \overline{A}BC + A \\
&= (\overline{A} + A)(BC + A) \\
&= 1 \cdot (BC + A)
\end{aligned}
$$

Combinational Circuits

Example (Combinational Circuit)

$$\begin{aligned}
\overline{A}BC + A\overline{BC} + A\overline{B}C + AB\overline{C} + ABC \\
&= \overline{A}BC + A\overline{B}(\overline{C} + C) + AB(\overline{C} + C) \\
&= \overline{A}BC + A\overline{B} \cdot 1 + AB \cdot 1 \\
&= \overline{A}BC + A\overline{B} + AB \\
&= \overline{A}BC + A(\overline{B} + B) \\
&= \overline{A}BC + A \cdot 1 \\
&= \overline{A}BC + A \\
&= (\overline{A} + A)(BC + A) \\
&= 1 \cdot (BC + A) \\
&= BC + A.
\end{aligned}$$

Example (Combinational Circuit)

$$\overline{ABC} + \overline{AB}C + \overline{A}B\overline{C}$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuit)

$$\overline{ABC} + \overline{AB}C + \overline{A}B\overline{C} = \overline{A}(\overline{BC} + \overline{B}C + B\overline{C})$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuit)

$$\overline{ABC} + \overline{AB}C + \overline{A}B\overline{C} = \overline{A}(\overline{BC} + \overline{B}C + B\overline{C})$$

$$= \overline{A}(\overline{BC} + \overline{B}C + B\overline{C} + \overline{BC})$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuit)

$$\overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}B\overline{C} = \overline{A}(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C})$$
$$= \overline{A}(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C} + \overline{B}\,\overline{C})$$
$$= \overline{A}(\overline{B}(\overline{C} + C) + (B + \overline{B})\overline{C})$$

- Or we could find the complementary expression and then invert it.

Combinational Circuits

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C + \overline{A}B\overline{C} &= \overline{A}(\overline{B}\overline{C} + \overline{B}C + B\overline{C}) \\
&= \overline{A}(\overline{B}\overline{C} + \overline{B}C + B\overline{C} + \overline{B}\overline{C}) \\
&= \overline{A}(\overline{B}(\overline{C} + C) + (B + \overline{B})\overline{C}) \\
&= \overline{A}(\overline{B} + \overline{C}).
\end{aligned}
$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuit)

$$\begin{aligned}
\overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}B\overline{C} &= \overline{A}(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C}) \\
&= \overline{A}(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C} + \overline{B}\,\overline{C}) \\
&= \overline{A}(\overline{B}(\overline{C} + C) + (B + \overline{B})\overline{C}) \\
&= \overline{A}(\overline{B} + \overline{C}).
\end{aligned}$$

$$\overline{\overline{A}(\overline{B} + \overline{C})}$$

- Or we could find the complementary expression and then invert it.

Combinational Circuits

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}B\overline{C} &= \overline{A}(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C}) \\
&= \overline{A}(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C} + \overline{B}\,\overline{C}) \\
&= \overline{A}(\overline{B}(\overline{C} + C) + (B + \overline{B})\overline{C}) \\
&= \overline{A}(\overline{B} + \overline{C}). \\
\overline{\overline{A}(\overline{B} + \overline{C})} &= \overline{\overline{A}} + \overline{(\overline{B} + \overline{C})}
\end{aligned}
$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{ABC} + \overline{AB}C + \overline{A}B\overline{C} &= \overline{A}(\overline{BC} + \overline{B}C + B\overline{C}) \\
&= \overline{A}(\overline{BC} + \overline{B}C + B\overline{C} + \overline{BC}) \\
&= \overline{A}(\overline{B}(\overline{C} + C) + (B + \overline{B})\overline{C}) \\
&= \overline{A}(\overline{B} + \overline{C}). \\
\overline{\overline{A}(\overline{B} + \overline{C})} &= \overline{\overline{A}} + \overline{(\overline{B} + \overline{C})} \\
&= A + \overline{\overline{B}} \cdot \overline{\overline{C}}
\end{aligned}
$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuit)

$$
\begin{aligned}
\overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C + \overline{A}B\overline{C} &= \overline{A}(\overline{B}\overline{C} + \overline{B}C + B\overline{C}) \\
&= \overline{A}(\overline{B}\overline{C} + \overline{B}C + B\overline{C} + \overline{B}\overline{C}) \\
&= \overline{A}(\overline{B}(\overline{C} + C) + (B + \overline{B})\overline{C}) \\
&= \overline{A}(\overline{B} + \overline{C}). \\
\overline{\overline{A}(\overline{B} + \overline{C})} &= \overline{\overline{A}} + \overline{(\overline{B} + \overline{C})} \\
&= A + \overline{\overline{B}} \cdot \overline{\overline{C}} \\
&= A + BC.
\end{aligned}
$$

- Or we could find the complementary expression and then invert it.

Example (Combinational Circuits)

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Find a boolean expression for this circuit.

Example (Combinational Circuits)

A	B	C	Output
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Find a boolean expression for this circuit.

Outline

Assignment

Assignment

- Read Section C.2 (on CD).