

Banzhaf Power

Lecture 13
Section 2.2

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- 1 Coalitions
- 2 Critical Players
- 3 The Banzhaf Power Index
- 4 Examples
- 5 Assignment

Outline

- 1 Coalitions
- 2 Critical Players
- 3 The Banzhaf Power Index
- 4 Examples
- 5 Assignment

Coalitions

Definition (Coalition)

A **coalition** is a group of players who agree to vote as a block. A **winning coalition** is a coalition whose votes add up to at least the quota. A **losing coalition** is a coalition whose votes add up to less than the quota.

Number of Coalitions

- If there are 3 players, how many possible coalitions are there?

Number of Coalitions

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- How about 4 players?

Number of Coalitions

- If there are 3 players, how many possible coalitions are there?
- How about 4 players?
- 5 players?

Number of Coalitions

Listing Coalitions

(Nobody)

A

B

AB

2 Players, 4 coalitions

Number of Coalitions

Listing Coalitions

(Nobody)

A

B

C

AB

AC

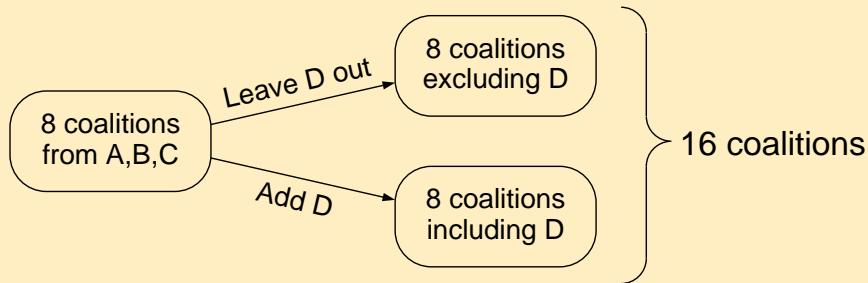
BC

ABC

3 Players, 8 coalitions

Number of Coalitions

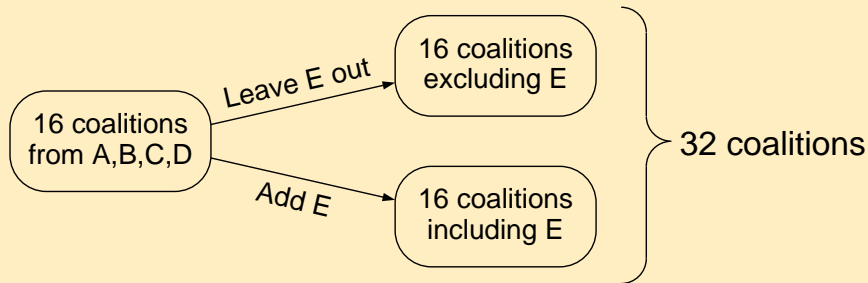
Listing Coalitions



4 Players, 16 coalitions

Number of Coalitions

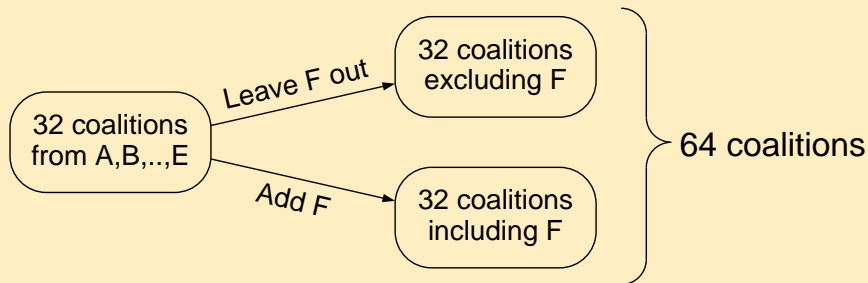
Listing Coalitions



5 Players, 32 coalitions

Number of Coalitions

Listing Coalitions



6 Players, 64 coalitions

Number of Coalitions

0	1	2	3	4	5	6
\emptyset	A	AB	ABC	ABCD	ABCDE	ABCDEF
	B	AC	ABD	ABCE	ABCDF	
	C	AD	ABE	ABCF	ABCEF	
	D	AE	ABF	ABDE	ABDEF	
	E	AF	ACD	ABDF	ACDEF	
	F	BC	ACE	ABEF	BCDEF	
		BD	ACF	ACDE		
		BE	ADE	ACDF		
		BF	ADF	ACEF		
		CD	AEF	ADEF		
		CE	BCD	BCDE		
		CF	BCE	BCDF		
		DE	BCF	BCEF		
		DF	BDE	BDEF		
		EF	BDF	CDEF		
			BEF			
			CDE			
			CDF			
			CEF			
			DEF			

Number of Coalitions

- Everytime we add one more player, the number of coalitions doubles (counting the empty coalition).
- Thus, if there are N players, then there are 2^N coalitions.
- What if there were 15 players?

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- Thus, if there are N players, then there are 2^N coalitions.
- What if there were 15 players?
- Then there would be $2^{15} = 32,768$ coalitions.
- If there were 25 players, there would be $2^{25} = 33,554,432$ coalitions.

Listing Coalitions

- The best way to list the possible coalitions is by size.
 - Start with the empty set (or skip it).
 - Consider all coalitions of a single player: A , B , C , ...
 - Then consider all coalitions of two players by adding a player to the singleton coalitions: AB , AC , BC , ...
 - Then coalitions of three players, then four players, and so on.

Listing Coalitions By Size

By Size

\emptyset

The empty coalition

Listing Coalitions By Size

By Size

\emptyset

A

B

C

Coalitions of size 1

Listing Coalitions By Size

By Size

	A	AB
\emptyset	B	AC
	C	BC

Coalitions of size 2

Listing Coalitions By Size

By Size

	A	AB	
\emptyset	B	AC	ABC
	C	BC	

Coalition of size 3

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Critical Players

Definition (Critical Player)

A **critical player** of a coalition is a player whose membership in that coalition takes it from a losing coalition to a winning coalition. That is,

- When that player is **included**, it is a **winning** coalition, but
- When that player is **excluded**, it is a **losing** coalition.

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
<i>A</i>		
<i>B</i>		
<i>C</i>		
<i>AB</i>		
<i>AC</i>		
<i>BC</i>		
<i>ABC</i>		

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
<i>A</i>	4	
<i>B</i>	2	
<i>C</i>	1	
<i>AB</i>	6	
<i>AC</i>	5	
<i>BC</i>	3	
<i>ABC</i>	7	

An Example

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<i>A</i>	4	
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<i>AB</i>	6	<i>A, B</i>
<i>AC</i>	5	<i>A, C</i>
<i>BC</i>	3	
<i>ABC</i>	7	<i>A</i>

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<i>AB</i>	6	<i>A, B</i>
<i>AC</i>	5	<i>A, C</i>
<i>BC</i>	3	
<i>ABC</i>	7	<i>A</i>

Notice that *A* has veto power, but is not a dictator.

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
A	4	
B	2	
C	1	
AB	6	A, B
AC	5	A, C
BC	3	
ABC	7	A

Notice that A has veto power, but is not a dictator. How can we tell?

Veto Power

Veto Power

A player has veto power if he is a member of *every winning coalition*.

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
<i>A</i>		
<i>B</i>		
<i>C</i>		
<i>AB</i>		
<i>AC</i>		
<i>BC</i>		
<i>ABC</i>		

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
<i>A</i>	4	
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What if the quota were lowered to 4?

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<i>A</i>	4	
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What if the quota were lowered to 4?

Coalition	Weight	Critical Players
<i>A</i>	4	<i>A</i>
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<i>AB</i>	6	<i>A</i>
<i>AC</i>	5	<i>A</i>
<i>BC</i>	3	
<i>ABC</i>	7	

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Now *A* is a dictator and has veto power.

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
<i>A</i>	4	<i>A</i>
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<i>AB</i>	6	<i>A</i>
<i>AC</i>	5	<i>A</i>
<i>BC</i>	3	
<i>ABC</i>	7	

Now *A* is a dictator and has veto power. How can we tell?

Dictator

Dictator

A player A is a dictator if the “coalition” A is a winning coalition.

Dictators and Veto Power

- Can a player A be a dictator and not have veto power?

Dictators and Veto Power

- Can a player A be a dictator and not have veto power?
- That is, can $\{A\}$ be a winning coalition, yet A not be a critical player in every winning coalition?

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- That is, can $\{A\}$ be a winning coalition, yet A not be a critical player in every winning coalition?
- Yes, but only if we have anarchy!

Dictators and Veto Power

- Can a player A be a dictator and not have veto power?
- That is, can $\{A\}$ be a winning coalition, yet A not be a critical player in every winning coalition?
- Yes, but only if we have anarchy!
- Consider $[4 : 4, 3, 1]$.

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
$\{A\}$		
$\{B\}$		
$\{C\}$		
$\{D\}$		
$\{A, B\}$		
$\{A, C\}$		
$\{A, D\}$		
$\{B, C\}$		
$\{B, D\}$		
$\{C, D\}$		
$\{A, B, C\}$		
$\{A, B, D\}$		
$\{A, C, D\}$		
$\{B, C, D\}$		
$\{A, B, C, D\}$		

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
{A}	9	
{B}	8	
{C}	3	
{D}	1	
{A, B}	17	
{A, C}	12	
{A, D}	10	
{B, C}	11	
{B, D}	9	
{C, D}	4	
{A, B, C}	20	
{A, B, D}	18	
{A, C, D}	13	
{B, C, D}	12	
{A, B, C, D}	21	

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
$\{A\}$	9	
$\{B\}$	8	
$\{C\}$	3	
$\{D\}$	1	
$\{A, B\}$	17	A, B
$\{A, C\}$	12	A, C
$\{A, D\}$	10	
$\{B, C\}$	11	B, C
$\{B, D\}$	9	
$\{C, D\}$	4	
$\{A, B, C\}$	20	(none)
$\{A, B, D\}$	18	A, B
$\{A, C, D\}$	13	A, C
$\{B, C, D\}$	12	B, C
$\{A, B, C, D\}$	21	(none)

Dictators and Veto Power

Dictators and Veto Power

In the last example,

- Is there a dictator?

Dictators and Veto Power

Dictators and Veto Power

In the last example,

- Is there a dictator? **No**

Dictators and Veto Power

In the last example,

- Is there a dictator? **No**
- Does anyone have veto power?

Dictators and Veto Power

Dictators and Veto Power

In the last example,

- Is there a dictator? **No**
- Does anyone have veto power? **No**

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Definitions

Definition (Critical Count)

The **critical count** of a player is the number of possible coalitions in which he is a critical player.

Definition (Banzhaf Power Index)

The **Banzhaf power index (BPI)** of a player is that player's critical count divided by the total of all players' critical counts.

Definition (Banzhaf Power Distribution)

The **Banzhaf power distribution** is the set of BPI's for all the players.

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Example

Example

- Find the power distribution in $[11 : 9, 8, 3, 1]$.

Example

Example

- Find the power distribution in $[11 : 9, 8, 3, 1]$.
- Does this sound right?

Example

Example

- Find the power distribution in $[9 : 5, 4, 3, 2, 1]$.
- You are E and you would like to buy one vote from another player. From which player should you buy it?

Example

Example

- Consider the situation $[q : 3, 3, 2, 1]$.
- What quota(s) q makes the power distribution most balanced?
- What quota(s) q makes the power distribution most unbalanced?

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Assignment

Assignment

- Chapter 2: Exercises 11, 12, 13, 14, 15, 17, 19; 69, 71. (You may want to use the Javascript program for 69 and 71.)