The Divider-Chooser Method

Lecture 15
Sections 3.1 - 3.2

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Outline

Algorithms

Definition (Algorithm)

An algorithm to solve a problem is a step-by-step procedure that is guaranteed to terminate with the correct answer to that problem.

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  - Blackjack player: “My total so far is 14. Should I say ‘hit’ or should I say ‘stand?’ ” “I think I’ll say ‘stand’.” “No, maybe I’ll say ‘hit’.”
- An algorithm may involve *straightforward* (i.e., rule-based) *decisions*.
  - Blackjack player: “Rule: if my total is at least 16, then will say ‘stand.’ Otherwise, I will say ‘hit.’ ”

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An algorithm to solve a problem is a step-by-step procedure that is guaranteed to terminate with the correct answer to that problem.

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  - Blackjack player: “My total so far is 14. Should I say ‘hit’ or should I say ‘stand?’ ” “I think I’ll say ‘stand’.” “No, maybe I’ll say ‘hit’.”
- An algorithm may involve *straightforward* (i.e., rule-based) *decisions*.
  - Blackjack player: “Rule: if my total is at least 16, then will say ‘stand.’ Otherwise, I will say ‘hit.’ ” “My total is 14, so I will say ‘hit.’ ”

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An Example

Example (Dividing Pies)

- Suppose Andy and Bob have three pies to share: 2 apple pies and one cherry pie.

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- Would it be optimal?

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- Would it be optimal? No

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- What would be a very simple method to divide the pies fairly? Cut each pie in half.
- What if Andy prefers apple pie “twice as much” as cherry and Bob prefers cherry “twice as much” as apple? Would that method still be fair? Yes
- Would it be optimal? No
- Why not?

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Example (Dividing Pies)

- Let's say that Andy values each apple pie at $2.00 and the cherry pie at $1.00 and Bob values each apple pie at $3.00 and the cherry pie at $6.00?

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- If each gets one apple pie (two halves) and half of the cherry pie, then Andy values his share at \$2.50 and Bob values his share at \$6.00.

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- Let's say that Andy values each apple pie at $2.00 and the cherry pie at $1.00 and Bob values each apple pie at $3.00 and the cherry pie at $6.00?
- If each gets one apple pie (two halves) and half of the cherry pie, then Andy values his share at $2.50 and Bob values his share at $6.00.
- That is fair (why?), but it is not optimal.
- What solution would be both fair (whatever that means) and optimal (whatever that means)?

Outline

Assets and Players

Definition (The Assets)

The assets are the physical objects to be divided. Let S denote the set of assets.

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The players are those among whom the assets are to be divided. Let N denote the number of players.

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- We assume in this chapter that the assets are "infinitely" divisible.

Value Systems and Fair Division

Definition (The Value Systems)

The value system of a player is the set of values assigned to the individual assets by that player.

- There is a separate value system for each player.
- No player knows the value system of any other player.

Value Systems and Fair Division

Definition (Fair Share)

A player's fair share is the fraction $\frac{1}{N}$ of the total value of the assets, *according to that player's value system* (where N is the number of players).

Definition (Fair Division)

A fair division is a division of the assets in which each player gets *at least* his fair share.

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A fair division is a division of the assets in which each player gets *at least* his fair share.

- Except in special cases (see next slide), we will *never* simply cut each asset into equal parts.

Special Cases

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- (**Only one asset**) If there is one pie to divide between Andy and Bob, then the only solution is to cut the pie in half.
- (**Identical value systems**) If there are several pies, and Andy and Bob have identical value systems, then we can do no better than to cut each pie in half.

Special Cases

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- (**Identical value systems**) If there are several pies, and Andy and Bob have identical value systems, then we can do no better than to cut each pie in half.

- Only in these cases may we divide each asset into equal parts.

Outline

Definition (The Divider-Chooser Method (2 Players))

The divider-chooser method involves exactly 2 players. One player is selected (arbitrarily) to be the divider. The other player is the chooser.

- The **divider divides** the assets into two equal shares, according to the divider's value system.
- The **chooser chooses** the share that he prefers according to the chooser's value system.
- The divider gets the other share.

Outline

Example

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Suppose that Andy and Bob will share four pies. Their value systems are shown in the following table.

	Apple	Cherry	Lemon	Pecan
Andy	10	8	6	4
Bob	6	4	8	2

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- If Andy is the divider, how should he divide the pies? S_1: A + P; S_2: C + L
- How should Bob choose? S_2
- Final division: Andy gets A + P ($14.00); Bob gets C + L ($12.00).

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	Apple	Cherry	Lemon	Pecan
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- If Bob is the divider, how should he divide the pies? S_1: A + C; S_2: L + P
- How should Andy choose? S_1
- Final division: Andy gets A + C ($18.00); Bob gets L + P ($10.00).

Example

Suppose that Andy and Bob will share four pies. Their value systems are shown in the following table.

	Apple	Cherry	Lemon	Pecan
Andy	4	6	2	2
Bob	6	4	3	5

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- If Andy is the divider, how should he divide the pies? S_1: C + $\frac{1}{2}$L; S_2: A + $\frac{1}{2}$L + P

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- Which should Bob choose? S_2
- Final division: Andy gets C + $\frac{1}{2}$L (\$7.00); Bob gets A + $\frac{1}{2}$L + P (\$12.50).

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- Which should Andy choose? S_2
- Final division: Andy gets C + P ($8.00); Bob gets A + L ($9.00).

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- How should Bob choose? S_2
- Final division: Andy gets A + $\frac{1}{4}$C ($12.00); Bob gets $\frac{3}{4}$C + L ($11.00).

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- If Bob is the divider, how should he divide the pies? S_1: A + $\frac{3}{4}$C; S_2: $\frac{1}{4}$C + L
- How should Andy choose? S_1
- Final division: Andy gets A + $\frac{3}{4}$C ($16.00); Bob gets $\frac{1}{4}$C + L ($9.00).

Outline

Assignment

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- Chapter 3: Exercises 1, 3, 5, 15, 16, 17, 18, 19, 20.