The Payoff Matrix

Lecture 35

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Outline

Pure Strategies

Definition (Strategy)

A strategy is a rule that tells the player which options to choose.

Definition (Pure Strategy)

A pure strategy is a strategy in which the player always chooses the same option.

Definition (Optimal Strategy)

An optimal strategy is a strategy that produces the best outcome possible for the player, given that the other player will also use his optimal strategy.

- Later we will introduce mixed strategies.

The Prisoners' Dilemma

The Prisoners' Dilemma

		Bob	
		Cooperate	Not Cooperate
Andy	Cooperate	$(3, 3)$	$(1, 4)$
	Not Cooperate	$(4, 1)$	$(2, 2)$

The Prisoners' Dilemma

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		Bob	
		Cooperate	Not Cooperate
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- Each player's optimal pure strategy is to cooperate.

The Prisoners' Dilemma

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		Bob	
		Cooperate	Not Cooperate
Andy	Cooperate	$(3,3)$	$(1,4)$
	Not Cooperate	$(4,1)$	$(2,2)$

- Each player's optimal pure strategy is to cooperate.
- The pure strategy "Not Cooperate" is not optimal, given that the other player will cooperate.

Example with 3 Options

- Andy and Bob are playing a game where Andy picks a number from 1 to 3 and Bob tries to guess the number. The payoffs are as follows.
  - (i) If Bob guesses correctly, then Bob wins the number guessed, in dollars ($1, $2, or $3).
  - (ii) If Bob guesses a larger number than the one Andy chose, then Andy wins the sum of the two numbers, in dollars.
  - (iii) If Bob guesses a smaller number than the one Andy chose, then Bob wins twice the sum of the two numbers.
- Then
  - (a) List the nine possible outcomes and rank them for each player.
  - (b) Draw the game matrix, including the ranks.
  - (c) Decide what each player will do.

Example with 3 Options

Example with 3 Options

Andy

Outcome	Payoff	Rank
$(1,1)$		
$(1,2)$		
$(1,3)$		
$(2,1)$		
$(2,2)$		
$(2,3)$		
$(3,1)$		
$(3,2)$		
$(3,3)$		

Bob

Outcome	Payoff	Rank
$(1,1)$		
$(1,2)$		
$(1,3)$		
$(2,1)$		
$(2,2)$		
$(2,3)$		
$(3,1)$		
$(3,2)$		
$(3,3)$		

Example with 3 Options

Example with 3 Options

Andy

Outcome	Payoff	Rank
$(1,1)$	-1	
$(1,2)$		
$(1,3)$		
$(2,1)$		
$(2,2)$	-2	
$(2,3)$		
$(3,1)$		
$(3,2)$		
$(3,3)$	-3	

Bob

Outcome	Payoff	Rank
$(1,1)$	$+1$	
$(1,2)$		
$(1,3)$		
$(2,1)$		
$(2,2)$	$+2$	
$(2,3)$		
$(3,1)$		
$(3,2)$		
$(3,3)$	$+3$	

Example with 3 Options

Example with 3 Options

Andy

Outcome	Payoff	Rank
$(1,1)$	-1	
$(1,2)$	$+3$	
$(1,3)$	$+4$	
$(2,1)$		
$(2,2)$	-2	
$(2,3)$	$+5$	
$(3,1)$		
$(3,2)$		
$(3,3)$	-3	

Bob

Outcome	Payoff	Rank
$(1,1)$	$+1$	
$(1,2)$	-3	
$(1,3)$	-4	
$(2,1)$		
$(2,2)$	$+2$	
$(2,3)$	-5	
$(3,1)$		
$(3,2)$		
$(3,3)$	$+3$	

Example with 3 Options

Andy

Outcome	Payoff	Rank
$(1,1)$	-1	
$(1,2)$	$+3$	
$(1,3)$	$+4$	
$(2,1)$	-6	
$(2,2)$	-2	
$(2,3)$	$+5$	
$(3,1)$	-8	
$(3,2)$	-10	
$(3,3)$	-3	

Bob

Outcome	Payoff	Rank
$(1,1)$	$+1$	
$(1,2)$	-3	
$(1,3)$	-4	
$(2,1)$	$+6$	
$(2,2)$	$+2$	
$(2,3)$	-5	
$(3,1)$	$+8$	
$(3,2)$	$+10$	
$(3,3)$	$+3$	

Example with 3 Options

Example with 3 Options

Andy

Outcome	Payoff	Rank
$(1,1)$	-1	4
$(1,2)$	$+3$	3
$(1,3)$	$+4$	2
$(2,1)$	-6	7
$(2,2)$	-2	5
$(2,3)$	$+5$	1
$(3,1)$	-8	8
$(3,2)$	-10	9
$(3,3)$	-3	6

Bob

Outcome	Payoff	Rank
$(1,1)$	$+1$	
$(1,2)$	-3	
$(1,3)$	-4	
$(2,1)$	$+6$	
$(2,2)$	$+2$	
$(2,3)$	-5	
$(3,1)$	$+8$	
$(3,2)$	$+10$	
$(3,3)$	$+3$	

Example with 3 Options

Andy

Outcome	Payoff	Rank
$(1,1)$	-1	4
$(1,2)$	$+3$	3
$(1,3)$	$+4$	2
$(2,1)$	-6	7
$(2,2)$	-2	5
$(2,3)$	$+5$	1
$(3,1)$	-8	8
$(3,2)$	-10	9
$(3,3)$	-3	6

Bob

Outcome	Payoff	Rank
$(1,1)$	$+1$	6
$(1,2)$	-3	7
$(1,3)$	-4	8
$(2,1)$	$+6$	3
$(2,2)$	$+2$	5
$(2,3)$	-5	9
$(3,1)$	$+8$	2
$(3,2)$	$+10$	1
$(3,3)$	$+3$	4

Example with 3 Options

Example with 3 Options

		Bob		
		1	2	3
Andy	1	$(4, 6)$	$(3, 7)$	$(2, 8)$
	2	$(7, 3)$	$(5, 5)$	$(1, 9)$
	3	$(8, 2)$	$(9, 1)$	$(6, 4)$

- What will Andy and Bob do?

Example with 3 Options

Example with 3 Options

		Bob		
		1	2	3
Andy	1	$(4, 6)$	$(3, 7)$	$(2, 8)$
	2	$(7, 3)$	$(5, 5)$	$(1, 9)$
	3	$(8, 2)$	$(9, 1)$	$(6, 4)$

- What will Andy and Bob do?
- Clearly, Andy will not choose 3.

Example with 3 Options

Example with 3 Options

		Bob		
		1	2	3
Andy	1	$(4,6)$	$(3,7)$	$(2,8)$
	2	$(7,3)$	$(5,5)$	$(1,9)$

- What will Andy and Bob do?
- Clearly, Andy will not choose 3.

Example with 3 Options

Example with 3 Options

		Bob		
		1	2	3
Andy	1	$(4, 6)$	$(3, 7)$	$(2, 8)$
	2	$(7, 3)$	$(5, 5)$	$(1, 9)$

- What will Andy and Bob do?
- Clearly, Andy will not choose 3.
- And clearly, Bob will not choose 3.

Example with 3 Options

Example with 3 Options

		Bob		
		1	2	
	1	$(4, 6)$	$(3, 7)$	
Andy	2	$(7, 3)$	$(5, 5)$	

- What will Andy and Bob do?
- Clearly, Andy will not choose 3.
- And clearly, Bob will not choose 3.

Example with 3 Options

Example with 3 Options

		Bob	
		1	2
Andy	1	$(4, 6)$	$(3, 7)$
	2	$(7, 3)$	$(5, 5)$

- So, for all practical purposes, Andy and Bob have only 2 options: 1 or 2.
- What will Andy and Bob do?

Example with 3 Options

		Bob	
		1	2
Andy	1	$(4, 6)$	$(3, 7)$
	2	$(7, 3)$	$(5, 5)$

- So, for all practical purposes, Andy and Bob have only 2 options: 1 or 2.
- What will Andy and Bob do?
- Clearly, Andy will not choose 2.

Example with 3 Options

Example with 3 Options

		Bob	
		1	2
Andy	1	$(4, 6)$	$(3, 7)$

- So, for all practical purposes, Andy and Bob have only 2 options: 1 or 2.
- What will Andy and Bob do?
- Clearly, Andy will not choose 2.

Example with 3 Options

Example with 3 Options

		Bob	
		1	2
Andy	1	$(4, 6)$	$(3, 7)$

- So, for all practical purposes, Andy and Bob have only 2 options: 1 or 2.
- What will Andy and Bob do?
- Clearly, Andy will not choose 2.
- And clearly, Bob will not choose 2.

Example with 3 Options

Example with 3 Options

Bob

		1	
Andy	1	$(4,6)$	

- So, for all practical purposes, Andy and Bob have only 2 options: 1 or 2.
- What will Andy and Bob do?
- Clearly, Andy will not choose 2.
- And clearly, Bob will not choose 2.

Example with 3 Options

Example with 3 Options

		Bob
		1
Andy	1	$(4, 6)$

- Both players use their pure strategies of always choosing 1.

Example with 3 Options

Example with 3 Options

		Bob
		1
Andy	1	$(4, 6)$

- Both players use their pure strategies of always choosing 1.
- Bob wins $1 every time.

Example with 3 Options

Example with 3 Options

		Bob
		1
Andy	1	$(4, 6)$

- Both players use their pure strategies of always choosing 1.
- Bob wins $1 every time.
- This is not a very good game–Andy would refuse to play.

Example with 3 Options

Example with 3 Options

		Bob
		1
Andy	1	$(4, 6)$

- Both players use their pure strategies of always choosing 1.
- Bob wins \$1 every time.
- This is not a very good game–Andy would refuse to play, if he has any sense.

Outline

Fair Games

Definition (Payoff)

The payoff of an outcome is the value of that outcome to the player.

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The payoff of an outcome is the value of that outcome to the player.

Definition (Fair Games)

A fair game is a game in which the *average* payoff to each player is zero.

Definition (Zero-Sum Games)

A zero-sum game is a game in which whatever one player wins, the other player necessarily loses.

Zero-Sum Games

- The game in the previous example is not fair.

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- Typically, we want a game to be fair, or else one of the players would refuse to play.

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- Each player's ranking will be the reverse of the other player's ranking.

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- Typically, we want a game to be fair, or else one of the players would refuse to play.
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- Each player's ranking will be the reverse of the other player's ranking.
- Each payoff in a pair of payoffs is the negative of the other payoff.

Zero-Sum Games

- The game in the previous example is not fair.
- Typically, we want a game to be fair, or else one of the players would refuse to play.
- In a zero-sum game, there is no need to list pairs of rankings or payoffs.
- Each player's ranking will be the reverse of the other player's ranking.
- Each payoff in a pair of payoffs is the negative of the other payoff.
- Therefore, we will list only the payoff to the row player.

Outline

The Payoff Matrix

Definition (The Payoff Matrix)

The payoff matrix of a zero-sum game shows the payoffs to the row player. The payoff to the column player is always the negative of the payoff to the row player.

Prisoners' Dilemma Payoff Matrix

- In the Prisoners' Dilemma problem, suppose that the payoffs are in gained or lost income.
- Andy and Bob each figure they will lose $10,000 for each year in prison (can't rob banks).
- They each figure they will gain $10,000 for each year that they are free (rob one bank each year).
- What is the payoff matrix (in thousands of dollars)?

The Prisoners' Dilemma

The Prisoners' Dilemma

		Bob: Cooperate	Bob: Not Cooperate
Andy	Cooperate	$(-10, -10)$	$(30, -30)$
	Not Cooperate	$(-30, 30)$	$(10, 10)$

- This is *not* a zero-sum game.
- Based on their pure strategies, they will each have a net loss of \$10,000.

Example with 3 Options

Example with 3 Options

		Bob		
		1	2	3
Andy	1	-1	$+3$	$+4$
	2	-6	-2	$+5$
	3	-8	-10	-3

- Our example of Andy and Bob choosing a number from 1 to 3 is a zero-sum game.

Outline

A Card Game

A Card Game

- Andy is holding the 2 of spades (2♠) and the 5 of hearts (5♡).
- Bob is holding the 3 of clubs (3♣) and the 7 of diamonds (7♢).
- Each player plays one of his cards.
- If the colors match, the Andy wins the sum of the numbers.
- If the colors do not match, Bob player wins the sum of the numbers.
- Write the payoff matrix.

A Card Game

A Card Game

Bob

		3♣	7♢
Andy	2♠	$+5$	-9
	5♡	-8	$+12$

A Card Game

A Card Game

		Bob	
		3♣	7♢
Andy	2♠	+5	−9
	5♡	−8	+12

- It is clear that neither player has a pure strategy.

A Card Game

A Card Game

		Bob	
		3♣	7♢
Andy	2♠	+5	−9
	5♡	−8	+12

- It is clear that neither player has a pure strategy.
- We will deal with that in the next lecture.

A Card Game

- Andy is holding the 2 of spades ($2\spadesuit$), the 7 of hearts ($7\heartsuit$), and the king of diamonds ($K\diamondsuit$).
- Bob is holding the 3 of clubs ($3\clubsuit$), the 6 of diamonds ($6\diamondsuit$), and the queen of spades ($Q\spadesuit$).
- Each player plays one of his cards.
- If both are face cards (queen or king), then Andy wins $10.
- If only one is a face card, then the one who played the face card wins $8.
- If neither is a face card, then whichever player played the larger number, wins the difference between the numbers, in dollars.

Another Card Game

Another Card Game

		Bob		
		3♣	6♢	Q♠
	2♠	-1	-4	$+8$
Andy	7♡	$+4$	$+1$	$+8$
	K♢	-8	-8	10

Another Card Game

Another Card Game

		Bob		
		3♣	6♢	Q♠
Andy	2♠	-1	-4	$+8$
	7♡	$+4$	$+1$	$+8$
	K♢	-8	-8	10

- Does either player have a pure strategy?

Another Card Game

Another Card Game

		Bob		
		3♣	6♢	Q♠
Andy	2♠	−1	−4	+8
	7♡	+4	+1	+8
	K♢	−8	−8	10

- Does either player have a pure strategy?
- What should Andy and Bob do?

Another Card Game

Another Card Game

Bob

		3♣		6♢		Q♠
	2♠	-1		-4		$+8$
Andy	7♡	$+4$		$+1$		$+8$
	K♢	-8		-8		10

Outline

Assignment

Assignment

- Work the problems on Handout #2.