# Drawing down an Annuity

## Lecture 5

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# Outline

# Annuity Formula (Drawing Down)

- When $k$ is greater than one, then the formula is a bit more complicated.

$$M = P\left(\frac{r/k}{1-\left(1+\frac{r}{k}\right)^{-kt}}\right),$$

- Replace $r$ with $r/k$ and replace $t$ with $kt$.

# Annuity Formula (Drawing Down)

- If the withdrawals are annual, then $k = 1$ and the formula is

$$M = P\left(\frac{r}{1-(1+r)^{-t}}\right)$$

where $M$ is the amount withdrawn per period, $P$ is the amount in the annuity when the withdrawals begin, $r$ is the annual interest rate, and $t$ is the number of years..

- When $k$ is greater than one, then the formula is a bit more complicated.

$$M = P\left(\frac{r/k}{1-\left(1+\frac{r}{k}\right)^{-kt}}\right),$$

- Replace $r$ with $r/k$ and replace $t$ with $kt$.

# Example

## Example (Five Withdrawals)

- Suppose that a person has accumulated $10,000 and that it is earning 10% interest per year.
- How much can he withdraw each year for 5 years?

# Example

## Example (Five Withdrawals)

The amount withdrawn is

$$M = \frac{Pr}{1 - (1 + r)^{-t}}$$

# Example

## Example (Five Withdrawals)

The amount withdrawn is

$$M = \frac{Pr}{1-(1+r)^{-t}} = \frac{(10000)(.10)}{1-(1.10)^{-5}}$$

# Example

## Example (Five Withdrawals)

The amount withdrawn is

$$\begin{aligned} M &= \frac{Pr}{1-(1+r)^{-t}} \\ &= \frac{(10000)(.10)}{1-(1.10)^{-5}} \\ &= 2637.97. \end{aligned}$$

# Example

## Example (Three Withdrawals)

| Year | Starting Balance | Interest | Total | Withdrawal | Ending Balance |
|---|---|---|---|---|---|
| 1 | 10,000.00 | 1,000.00 | 11,000.00 | 2,637.97 | 8,362.03 |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

# Example

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|---|---|---|---|---|---|
| 1 | 10,000.00 | 1,000.00 | 11,000.00 | 2,637.97 | 8,362.03 |
| 2 | 8,362.03 | 836.20 | 9,198.23 | 2,637.97 | 6,560.26 |
| | | | | | |
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# Example

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| 2 | 8,362.03 | 836.20 | 9,198.23 | 2,637.97 | 6,560.26 |
| 3 | 6,560.26 | 656.03 | 7,216.29 | 2,637.97 | 4,578.32 |
| | | | | | |
| | | | | | |

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| 3 | 6,560.26 | 656.03 | 7,216.29 | 2,637.97 | 4,578.32 |
| 4 | 4,578.32 | 457.83 | 5,036.15 | 2,637.97 | 2,398.18 |
| | | | | | |

# Example

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| Year | Starting Balance | Interest | Total | Withdrawal | Ending Balance |
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| 3 | 6,560.26 | 656.03 | 7,216.29 | 2,637.97 | 4,578.32 |
| 4 | 4,578.32 | 457.83 | 5,036.15 | 2,637.97 | 2,398.18 |
| 5 | 2,398.18 | 239.82 | 2,638.00 | 2,637.97 | 0.03 |

# Example

## Example (Five Withdrawals)

- How much interest was earned over the 5 years?

# Example

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$$\begin{aligned} \text{Interest} &= 5 \times 2{,}637.97 - 10{,}000 \\ &= 13{,}189.85 - 10{,}000 \\ &= \$3{,}189.85. \end{aligned}$$

# Outline

# Example

## Example (10-Year Example)

- Suppose we invest $200.00 each month at 9% for 18 years for a college savings account.
- Then we withdraw from the account a fixed amount (to be determined) each year for the next 4 years (tuition payments).

# Example

## Example (Building up the Annuity)

The future value is of the annuity is

$$F = \frac{P((1 + \frac{r}{12})^{12t} - 1)}{r/12}$$

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The future value is of the annuity is

$$F = \frac{P((1+\frac{r}{12})^{12t}-1)}{r/12}$$
$$= \frac{200((1.0075)^{216}-1)}{0.0075}$$

# Example

## Example (Building up the Annuity)

The future value is of the annuity is

$$\begin{aligned} F &= \frac{P((1+\frac{r}{12})^{12t}-1)}{r/12} \\ &= \frac{200((1.0075)^{216}-1)}{0.0075} \\ &= \$107{,}270.33 \end{aligned}$$

# Example

## Example (Drawing down the Annuity)

- Now we begin making withdrawals over the next 4 years.
- How much can we withdraw each year?

# Example

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- Now we begin making withdrawals over the next 4 years.
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$$M = \frac{Pr}{1 - (1 + r)^{-t}}$$

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- Now we begin making withdrawals over the next 4 years.
- How much can we withdraw each year?

$$M = \frac{Pr}{1-(1+r)^{-t}}$$
$$= \frac{(107{,}270.33)(0.09)}{1-(1.09)^{-4}}$$

# Example

## Example (Drawing down the Annuity)

- Now we begin making withdrawals over the next 4 years.
- How much can we withdraw each year?

$$\begin{aligned} M &= \frac{Pr}{1-(1+r)^{-t}} \\ &= \frac{(107{,}270.33)(0.09)}{1-(1.09)^{-4}} \\ &= \$33{,}100.99 \end{aligned}$$

# Example

## Example (Drawing down the Annuity)

- What if the interest rate were 10%?

# Example

## Example (Drawing down the Annuity)

- What if the interest rate were 10%? ans: $37,892.03

# Example

## Example (Drawing down the Annuity)

- What if the interest rate were 10%? ans: $37,892.03
- What if the interest rate were 12%?

# Example

## Example (Drawing down the Annuity)

- What if the interest rate were 10%? ans: $37,892.03
- What if the interest rate were 12%? ans: $49,902.76

# Outline

# Another Example

## Example

- That same person says, “But I think I’ll need $75,000 each year for tuition and I’m afraid that I will earn only 6% on the average.”
- How much should the person invest each month?

# Example

## Example (Drawing down the Annuity)

- We have to work the problem “backwards.”
- What must be the value of the annuity in order to withdraw $50,000 each year for 4 years?

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# Example

## Example (Drawing down the Annuity)

- We have to work the problem “backwards.”
- What must be the value of the annuity in order to withdraw \$50,000 each year for 4 years?

$$M = \frac{Pr}{1-(1+r)^{-t}}$$

$$75000 = \frac{P(0.06)}{1-(1.06)^{-4}}$$

# Example

## Example (Drawing down the Annuity)

- We have to work the problem “backwards.”
- What must be the value of the annuity in order to withdraw \$50,000 each year for 4 years?

$$M = \frac{Pr}{1-(1+r)^{-t}}$$

$$75000 = \frac{P(0.06)}{1-(1.06)^{-4}}$$

$$= P(0.28859149)$$

# Example

## Example (Drawing down the Annuity)

- We have to work the problem “backwards.”
- What must be the value of the annuity in order to withdraw \$50,000 each year for 4 years?

$$
\begin{aligned}
M &= \frac{Pr}{1-(1+r)^{-t}} \\
75000 &= \frac{P(0.06)}{1-(1.06)^{-4}} \\
&= P(0.28859149) \\
P &= \frac{75{,}000}{0.28859149}
\end{aligned}
$$

# Example

## Example (Drawing down the Annuity)

- We have to work the problem "backwards."
- What must be the value of the annuity in order to withdraw \$50,000 each year for 4 years?

$$
\begin{aligned}
M &= \frac{Pr}{1-(1+r)^{-t}} \\
75000 &= \frac{P(0.06)}{1-(1.06)^{-4}} \\
&= P(0.28859149) \\
P &= \frac{75{,}000}{0.28859149} \\
&= \$259{,}882.92
\end{aligned}
$$

# Example

## Example (Building up the Annuity)

- Now how much must be invested each month at 6% interest in order to have $259,882.92 after 18 years?

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- Now how much must be invested each month at 6% interest in order to have \$259,882.92 after 18 years?

$$F = \frac{P((1 + \frac{r}{12})^{12t} - 1)}{r/12}$$

# Example

## Example (Building up the Annuity)

- Now how much must be invested each month at 6% interest in order to have \$259,882.92 after 18 years?

$$F = \frac{P((1+\frac{r}{12})^{12t}-1)}{r/12}$$

$$259{,}882.92 = \frac{P(1.005^{216}-1)}{0.005}$$

# Example

## Example (Building up the Annuity)

- Now how much must be invested each month at 6% interest in order to have \$259,882.92 after 18 years?

$$F = \frac{P((1+\frac{r}{12})^{12t}-1)}{r/12}$$

$$259{,}882.92 = \frac{P(1.005^{216}-1)}{0.005}$$

$$= P(387.35319)$$

# Example

## Example (Building up the Annuity)

- Now how much must be invested each month at 6% interest in order to have \$259,882.92 after 18 years?

$$F = \frac{P((1+\frac{r}{12})^{12t}-1)}{r/12}$$

$$259{,}882.92 = \frac{P(1.005^{216}-1)}{0.005}$$

$$= P(387.35319)$$

$$P = \frac{259{,}882.92}{387.35319}$$

# Example

## Example (Building up the Annuity)

- Now how much must be invested each month at 6% interest in order to have \$259,882.92 after 18 years?

$$F = \frac{P((1+\frac{r}{12})^{12t}-1)}{r/12}$$

$$\begin{aligned} 259{,}882.92 &= \frac{P(1.005^{216}-1)}{0.005} \\ &= P(387.35319) \\ P &= \frac{259{,}882.92}{387.35319} \\ &= \$670.92 \end{aligned}$$

# Outline

# Assignment

## Assignment

- Annuity worksheet: 6 - 10.