The Central Limit Theorem
Sections 15.4, 15.5
Lecture 28

Robb T. Koether

Hampden-Sydney College

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Outline

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1. The Central Limit Theorem
2. Applications
3. Assignment

The Central Limit Theorem

Theorem (Central Limit Theorem)

Draw a simple random sample of size n from any population (whatsoever). Let μ and σ be the mean and standard deviation, respectively, of that population. If n is "large enough," then the sample mean $\overline{x}$, as a random variable, is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$. That is,

$$\overline{x} \text{ is approximately } N\left(\mu, \frac{\sigma}{\sqrt{n}}\right).$$

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$$\overline{x} \text{ is approximately } N\left(\mu, \frac{\sigma}{\sqrt{n}}\right).$$

- Generally, $n \geq 30$ is large enough.
- Although it is an approximation, it is an *exceedingly good* approximation for large sample sizes.

A Special Case

Theorem

If the population is normal with mean μ and standard deviation σ, then the sample mean $\overline{x}$ is normal as well, and has mean μ and standard deviation $\sigma/\sqrt{n}$. That is, if

$$x \text{ is } N(\mu, \sigma),$$

then

$$\overline{x} \text{ is } N(\mu, \sigma/\sqrt{n}).$$

The Central Limit Theorem

- The Central Limit Theorem allows us to “break free” from the requirement that we sample from a normal population.
- The Central Limit Theorem says that it does not matter whether the population is normal, provided the sample size is large enough.

Outline

Application

Example (IQ Scores)

- Assume that IQ scores have a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$.
- Suppose we take a simple random sample of 100 people and measure their IQ scores.
- What is the probability that a single, randomly selected IQ score is between 95 and 105?
- What is the probability that the sample mean IQ score is between 95 and 105?

Application

Example (IQ Scores)

- The single IQ score comes from a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$.
- The probability that the single IQ score is between 95 and 105 is

$$\begin{aligned} \texttt{normalcdf(95,105,100,15)} &= 0.2611 \\ &= 26.11\%. \end{aligned}$$

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$$\texttt{normalcdf(95,105,100,15)} = 0.2611$$
$$= 26.11\%.$$

- That is, $P(95 \leq x \leq 105) = 0.2611$.

Application

Example (IQ Scores)

- The sample mean $\overline{x}$ has a normal distribution with mean $\mu = 100$ and standard deviation $\sigma/\sqrt{100} = 1.5$.
- The probability that the sample mean IQ score is between 95 and 105 is

$$\texttt{normalcdf(95,105,100,1.5)} = 0.9991$$
$$= 99.91\%.$$

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- That is, $P(95 \le \overline{x} \le 105) = 0.9991$.

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$$\begin{aligned}\texttt{normalcdf(95,105,100,1.5)} &= 0.9991 \\ &= 99.91\%.\end{aligned}$$

- That is, $P(95 \leq \overline{x} \leq 105) = 0.9991$.
- Major, major, major difference!

Application

Example (Test Scores)

- A large population of test scores is strongly skewed towards the lower scores.
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- What if we took a random sample of 1000 test scores and asked the same question?

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- A large population of test scores is strongly skewed towards the lower scores.
- We know that the mean score is 78 and the standard deviation is 12.
- If we take a random sample of 100 test scores, what is the probability that the average will be between 75 and 80?
- What if we took a random sample of 1000 test scores and asked the same question?
- What if we did not know that the population mean was 78, but we obtained a sample mean of 77.6? What could we conclude about the population mean? (Assume that $\sigma = 12$.)

Application

Example (Household Income)

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- The mean household income is about $52,000 and the standard deviation is about $35,000.
- If we take a random sample of 500 households, what is the probability that the average will be greater than $55,000?
- What if we took a random sample of 1000 test scores and asked the same question?

Outline

Assignment

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- Read Sections 15.4, 15.5.
- Apply Your Knowledge: 6, 8, 9, 10, 12.
- Check Your Skills: 20, 21, 22, 23.
- Exercises 28, 29, 30, 31.