Exponential Functions

Lecture 33
Section 4.1

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Mon, Mar 27, 2017

Reminder

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- Test #3 is this Friday, March 31.

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- Be there.

Objectives

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- Learn (or review) the properties of exponential expressions.
- Learn the properties of exponential functions.
- Learn about the "natural" base.

Exponential Expressions and Functions

Definition (Integer Exponential Expression)

Let b be a positive real number and let n be a positive integer. Then

$$b^n = \underbrace{b \cdot b \cdot b \cdots b}_{n \text{ factors}}$$

Also, $b^0 = 1$ and

$$b^{-n} = \frac{1}{b^n}.$$

The number b is called the **base** of the expression. The number n is called the **exponent**.

Exponential Expressions and Functions

Definition (Rational Exponential Expression)

Let b be a positive real number and let $\frac{n}{m}$ be a rational number. Then

$$b^{n/m} = \left(\sqrt[m]{b}\right)^n = \sqrt[m]{b^n}.$$

Properties of Exponential Expressions

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- Quotients (same exponent): $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$.
- Powers: $(b^x)^y = b^{xy}$.

Definition (Exponential Function)

An **exponential function** is a function of the form

$$f(x) = b^x$$

for some base $b > 0$.

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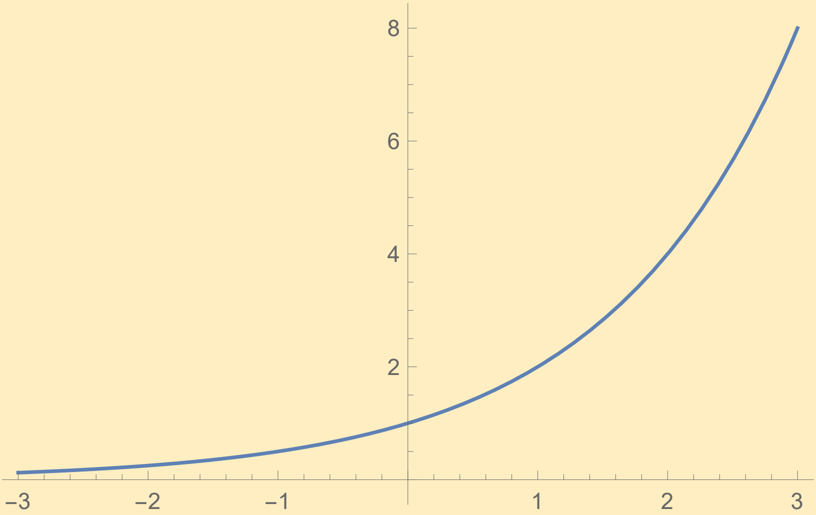
- If $b < 1$, then

$$\lim_{x \to +\infty} b^x = 0 \text{ and } \lim_{x \to -\infty} b^x = +\infty.$$

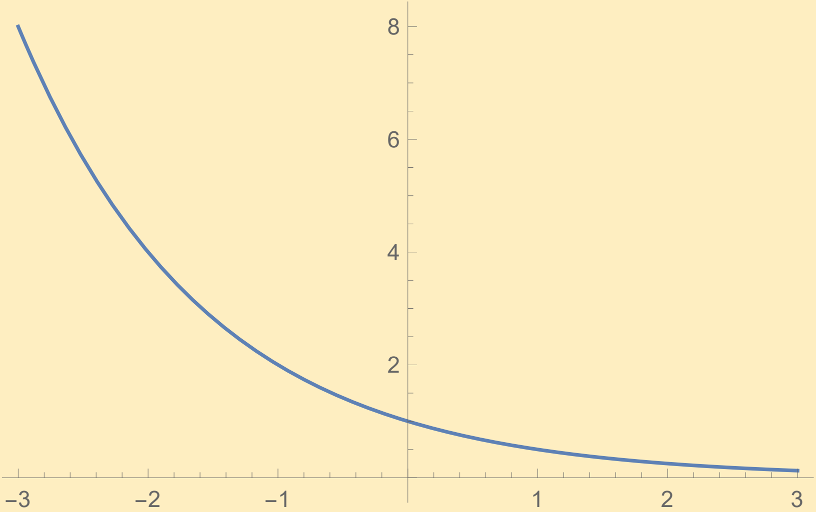
Graph of an Exponential Function ($b > 1$)

The graph of $f(x) = 2^x$



Graph of an Exponential Function ($b < 1$)

The graph of $f(x) = \left(\frac{1}{2}\right)^x$



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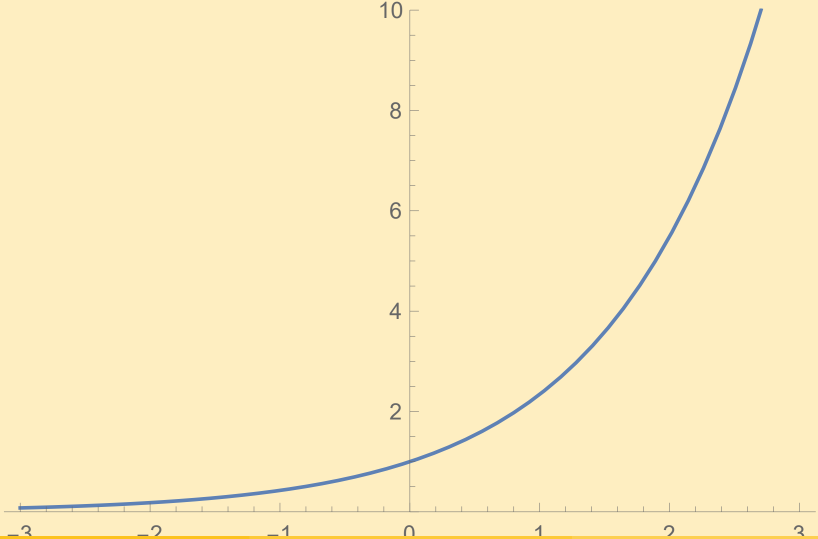

The Natural Base e

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- Let $f(x) = b^x$ for some base $b > 0$.
- What is $f'(x)$?

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- If we apply the definition, we get

$$f'(x) =$$

$$.$$

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$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

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$$= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}$$

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$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} \end{aligned}$$

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&= \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} \\
&= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} \\
&= \lim_{h\to 0} b^x \left(\frac{b^h-1}{h}\right)
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- Note that $\lim_{h \to 0} \frac{b^h - 1}{h}$ is a constant.
- For the right choice of b, that constant will be 1.
- That choice is approximately $2.71828\ldots$.
- We call that number e, the **natural base**.

Compound Interest

Let P be the present value of an investment, t the duration (in years) of the investment, r the annual interest rate, k the number of compounding periods per year, and $B(t)$ the future value after t years. Then

$$B(t) = P\left(1 + \frac{r}{k}\right)^{kt}.$$

Continuous Compounding

Continuous Compounding

If the interest is compounded continuously, then the future value is

$$B(t) = Pe^{rt}.$$