The Cobb-Douglas Production Functions

Lecture 40
Section 7.1

Robb T. Koether

Hampden-Sydney College

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Objectives

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- Define the Cobb-Douglas production functions.
- Explore their properties of scaling and elasticity.

The Cobb-Douglas Production Functions

Definition (The Cobb-Douglas Production Functions)

Let

- K the capital investment,
- L be the labor investment,
- Q be the output in units,
- A, α, and β be positive constants with $\alpha + \beta = 1$.

The **Cobb-Douglas production functions** are functions of the form

$$Q(K, L) = AK^{\alpha}L^{\beta}.$$

Constant Returns to Scale

Theorem

If $Q(K, L) = AK^\alpha L^\beta$, then the model predicts ***constant returns to scale****. That is, if capital investment and labor investment are both scaled by a factor s, then output will also scale by the factor s.*

Constant Returns to Scale

Proof.

- Replace K with sK and L with sL for some scale factor s.

□

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- Then

$$Q(sK, sL) = A(sK)^{\alpha}(sL)^{\beta}$$

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$$
\begin{aligned}
Q(sK, sL) &= A(sK)^{\alpha}(sL)^{\beta} \\
&= A(s^{\alpha}K^{\alpha})(s^{\beta}L^{\beta})
\end{aligned}
$$

□

Constant Returns to Scale

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$$\begin{aligned} Q(sK, sL) &= A(sK)^{\alpha}(sL)^{\beta} \\ &= A(s^{\alpha}K^{\alpha})(s^{\beta}L^{\beta}) \\ &= AK^{\alpha}L^{\beta}s^{\alpha+\beta} \end{aligned}$$

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$$\begin{aligned} Q(sK, sL) &= A(sK)^{\alpha}(sL)^{\beta} \\ &= A(s^{\alpha}K^{\alpha})(s^{\beta}L^{\beta}) \\ &= AK^{\alpha}L^{\beta}s^{\alpha+\beta} \\ &= s(AK^{\alpha}L^{\beta}) \\ &= sQ(K, L). \end{aligned}$$

□

Elasticity

Theorem

If $Q(K, L) = AK^{\alpha}L^{\beta}$, then

- α *is the capital investment* ***elasticity*** *of production, and*
- β *is the labor investment* ***elasticity*** *of production.*

Elasticity

Proof.

- Consider labor investment L to be held fixed, so that

$$Q(K) = AK^{\alpha}L^{\beta}.$$

- The formula for capital investment elasticity of production, in this case, is

$$E(Q) = \frac{K}{Q(K)} \cdot Q'(K).$$

- (We dropped the minus sign because production goes up, not down, as investment goes up.)

□

Elasticity

Proof.

- We compute the derivative $Q'(K) = A(\alpha K^{\alpha-1})L^{\beta}$.

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□

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Elasticity

Elasticity

- It is convenient, but not necessary, that $\alpha + \beta = 1$.

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- What behavior would you predict if $\alpha + \beta < 1$?

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- What behavior would you predict if $\alpha + \beta < 1$?
- What behavior would you predict if $\alpha + \beta > 1$?