Monday, September 14, 2015

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Problem 27

Problem. Write the expression $\sin \operatorname{arcsec} x$ in algebraic form.

Solution. Let $\theta = \operatorname{arcsec} x$. Then $\sec\theta = x$. Draw a right triangle with angle θ, hypotenuse x and adjacent side 1, so that $\sec\theta = x$. Then the opposite side is $\sqrt{x^2-1}$. It follows that $\sin\theta = \frac{\sqrt{x^2-1}}{x}$. So $\sin \operatorname{arcsec} x = \frac{\sqrt{x^2-1}}{x}$.

Problem 33

Problem. Solve the equation $\arcsin(3x-\pi) = \frac{1}{2}$ for x.

Solution.

$$\begin{aligned}
\arcsin(3x-\pi) &= \frac{1}{2}, \\
3x - \pi &= \sin\frac{1}{2}, \\
3x &= \pi + \sin\frac{1}{2}, \\
x &= \frac{1}{3}\left(\pi + \sin\frac{1}{2}\right).
\end{aligned}$$

Problem 35

Problem. Solve the equation $\arcsin\sqrt{2x} = \arccos\sqrt{x}$ for x.

Solution.

$$\begin{aligned}
\arcsin\sqrt{2x} = \arccos\sqrt{x}, & \\
\sqrt{2x} &= \sin\arccos\sqrt{x} \\
&= \sqrt{1-(\sqrt{x})^2} \\
&= \sqrt{1-x}, \\
2x &= 1-x, \\
3x &= 1, \\
x &= \frac{1}{3}.
\end{aligned}$$

Problem 41

Problem. Find the derivative of the function $g(x) = 3\arccos\frac{x}{2}$.

Solution. Use the differentiation formula for arccos and the Chain Rule.

$$\begin{aligned} g'(x) &= -3\sqrt{1-\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2} \\ &= -\frac{3}{2}\sqrt{1-\frac{x^2}{4}}. \end{aligned}$$

Problem 45

Problem. Find the derivative of the function $g(x) = \frac{\arcsin 3x}{x}$.

Solution. Use the differentiation formula for arcsin, the Chain Rule, and the Quotient Rule.

$$\begin{aligned} g'(x) &= \frac{\sqrt{1-(3x)^2}\cdot 3\cdot x - \arcsin 3x\cdot 1}{x^2} \\ &= \frac{3x\sqrt{1-9x^2} - \arcsin 3x}{x^2} \end{aligned}$$

Problem 49

Problem. Find the derivative of the function $y = 2x\arccos x - 2\sqrt{1-x^2}$.

Solution.

$$\begin{aligned} y' &= \left(2\cdot\arccos x + 2x\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)\right) - \left(2\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)\right) \\ &= 2\arccos x - \frac{2x}{\sqrt{1-x^2}} + \frac{2x}{\sqrt{1-x^2}} \\ &= 2\arccos x \end{aligned}$$

Problem 55

Problem. Find the derivative of the function $y = 8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$.

Solution.

$$\begin{aligned}
y' &= \frac{8}{\sqrt{1-\left(\frac{x}{4}\right)^2}} \cdot \frac{1}{4} - \frac{1}{2}\left(1 \cdot \sqrt{16-x^2} + x \cdot \frac{1}{2}(16-x^2)^{-1/2} \cdot (-2x)\right) \\
&= \frac{2}{\sqrt{1-\left(\frac{x}{4}\right)^2}} - \frac{1}{2}\left(\sqrt{16-x^2} - \frac{x^2}{\sqrt{16-x^2}}\right) \\
&= \frac{8}{\sqrt{16-x^2}} - \frac{1}{2}\left(\frac{16-x^2}{\sqrt{16-x^2}} - \frac{x^2}{\sqrt{16-x^2}}\right) \\
&= \frac{x^2}{\sqrt{16-x^2}}.
\end{aligned}$$

Problem 58

Problem. Find the derivative of the function $y = \arctan\frac{x}{2} - \frac{1}{2(x^2+4)}$.

Solution. Use the rule for arctan.

$$\begin{aligned}
y' &= \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} + \frac{2x}{2(x^2+4)^2} \\
&= \frac{2}{4+x^2} + \frac{x}{(x^2+4)^2} \\
&= \frac{2(4+x^2)}{(4+x^2)^2} + \frac{x}{(x^2+4)^2} \\
&= \frac{2x^2+x+8}{(4+x^2)^2}.
\end{aligned}$$

Problem 81

Problem. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

Solution. The trig functions all repeat their values over their domains. Therefore, none of them would pass the horizontal line test if their domains were not restricted. If they didn't pass the horizontal line test, then they would not have an inverse. It's as simple as that.

Problem 97

Problem. (a) Prove that $\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$.

(b) Use the formula in part (a) to show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

Solution. (a) Take the tangent of both sides and use the fact that

$$\tan\alpha + \beta = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

$$\begin{aligned}
\arctan x + \arctan y &= \arctan\frac{x+y}{1-xy} \\
\tan(\arctan x + \arctan y) &= \tan\left(\arctan\frac{x+y}{1-xy}\right) \\
\frac{\tan\arctan x + \tan\arctan y}{1 - \tan\arctan x \cdot \tan\arctan y} &= \frac{x+y}{1-xy} \\
\frac{x+y}{1-xy} &= \frac{x+y}{1-xy}.
\end{aligned}$$

(b) Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$. We get

$$\begin{aligned}
\arctan\frac{1}{2} + \arctan\frac{1}{3} &= \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) \\
&= \arctan\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right) \\
&= \arctan 1 \\
&= \frac{\pi}{4}.
\end{aligned}$$

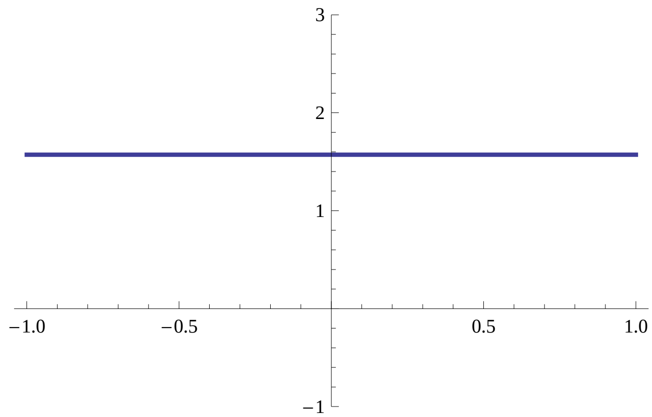
Problem 99

Problem. (a) Graph the function $f(x) = \arccos x + \arcsin x$ on the interval $[-1, 1]$.

(b) Describe the graph of f.

(c) Verify the result of part (b) analytically.

Solution. (a) The graph:



(b) It is just a horizontal line at approximately $y = 1.6$.

(c) Apply the sin function to $\arccos x + \arcsin x$ and use the identity

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$$\begin{aligned}\sin(\arccos x + \arcsin x) &= \sin\arccos x \cdot \cos\arcsin x + \cos\arccos x \cdot \sin\arcsin x \\ &= \sqrt{1 - x^2} \cdot \sqrt{1 - x^2} + x \cdot x \\ &= (1 - x^2) + x^2 \\ &= 1.\end{aligned}$$

Now take arcsin of both sides:

$$\arccos x + \arcsin x = \arcsin 1 = \frac{\pi}{2}.$$

Problem 101

Problem. In the figure (see the book), find the value of c in the interval $[0, 4]$ on the x-axis that maximizes the angle θ.

Solution. Express θ as a function of c. Let α be the angle to the left of θ in the diagram and let β be the angle to the right. Then we see that

$$\tan\alpha = \frac{2}{c}$$

and

$$\tan\beta = \frac{2}{4 - c}.$$

At this point, we suspect that it might be easier to use cotangent in order keep c in the numerator. That would be a good choice. So we rewrite these equations as

$$\cot\alpha = \frac{c}{2}$$

and

$$\cot\beta = \frac{4-c}{2}.$$

From the diagram, we see that $\alpha + \theta + \beta = \pi$. So

$$\begin{aligned}\theta &= \pi - \alpha - \beta \\ &= \pi - \operatorname{arccot}\frac{c}{2} - \operatorname{arccot}\left(\frac{4-c}{2}\right).\end{aligned}$$

Differentiate, set equal to 0, and solve (the usual procedure for maximizing a function).

$$\frac{d\theta}{dc} = \frac{1}{1+\left(\frac{c}{2}\right)^2}\cdot\frac{1}{2} + \frac{1}{1+\left(\frac{4-c}{2}\right)^2}\cdot\left(-\frac{1}{2}\right)$$

Solve $\frac{d\theta}{dc} = 0$.

$$\begin{aligned}\frac{1}{1+\left(\frac{c}{2}\right)^2}\cdot\frac{1}{2} + \frac{1}{1+\left(\frac{4-c}{2}\right)^2}\cdot\left(-\frac{1}{2}\right) &= 0, \\ \frac{1}{1+\left(\frac{c}{2}\right)^2} - \frac{1}{1+\left(\frac{4-c}{2}\right)^2} &= 0, \\ \frac{1}{1+\left(\frac{c}{2}\right)^2} &= \frac{1}{1+\left(\frac{4-c}{2}\right)^2}, \\ 1+\left(\frac{4-c}{2}\right)^2 &= 1+\left(\frac{c}{2}\right)^2, \\ \left(\frac{4-c}{2}\right)^2 &= \left(\frac{c}{2}\right)^2, \\ \frac{4-c}{2} &= \frac{c}{2}, \\ 4-c &= c, \\ 2c &= 4, \\ c &= 2.\end{aligned}$$

The angle θ is maximized when $c = 2$ (halfway between the two points).