Wednesday, September 30, 2015

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Problem 1

Problem. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the region $(y = x, y = 0, x = 2)$ about the y-axis.

Solution. The radius of a shell is $r = x$ and the height is $h = x$, so the volume is

$$\begin{aligned} V &= \int_0^2 2\pi x \cdot x \, dx \\ &= 2\pi \int_0^2 x^2 \, dx \\ &= 2\pi \left[\frac{1}{3} x^3 \right]_0^2 \\ &= \frac{16\pi}{3}. \end{aligned}$$

Problem 3

Problem. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the region $(y = 1 - x, y = 0, x = 0)$ about the y-axis.

Solution. The radius of a shell is $r = x$ and the height is $h = 1 - x$. The upper and lower boundaries meet at $x = 1$. The volume is

$$\begin{aligned} V &= \int_0^1 2\pi x(1 - x) \, dx \\ &= \pi \int_0^1 (2x - 2x^2) \, dx \\ &= \pi \left[x^2 - \frac{2}{3} x^3 \right]_0^1 \\ &= \pi \left(1 - \frac{2}{3} \right) \\ &= \frac{\pi}{3}. \end{aligned}$$

Problem 7

Problem. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the region bounded by

$$y = x^2,$$
$$y = 4x - x^2$$

about the y-axis.

The two curves meet at $x = 0$ and $x = \sqrt{2}$. The upper curve is $y = 4x - x^2$ and the lower curve is $y = x^2$. The radius is $r = x$ and the height is $h = (4x - x^2) - x^2 = 4x - 2x^2$. The volume is

$$\begin{aligned} V &= \int_0^{\sqrt{2}} 2\pi x(4x - 2x^2)\, dx \\ &= 2\pi \int_0^{\sqrt{2}} (4x^2 - 2x^3)\, dx \\ &= 2\pi \left[\frac{4}{3}x^3 - \frac{1}{2}x^4\right]_0^{\sqrt{2}} \\ &= 2\pi \left(\frac{8\sqrt{2}}{3} - 2\right) \\ &= \frac{(16\sqrt{2} - 12)\pi}{3}. \end{aligned}$$

Solution.

Problem 8

Problem. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the region bounded by

$$y = 9 - x^2,$$
$$y = 0$$

about the y-axis.

Solution. The upper curve meets the x-axis at $x = -3$ and $x = 3$. We should rotate the right half from $x = 0$ to $x = 3$. The height of the curve is $h = 9 - x^2$. The volume is

$$
\begin{aligned}
V &= \int_0^3 2\pi x(9 - x^2)\,dx \\
&= 2\pi \int_0^3 (9x - x^3)\,dx \\
&= 2\pi \left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3 \\
&= 2\pi \left(\frac{81}{2} - \frac{81}{4}\right) \\
&= \frac{81\pi}{2}.
\end{aligned}
$$

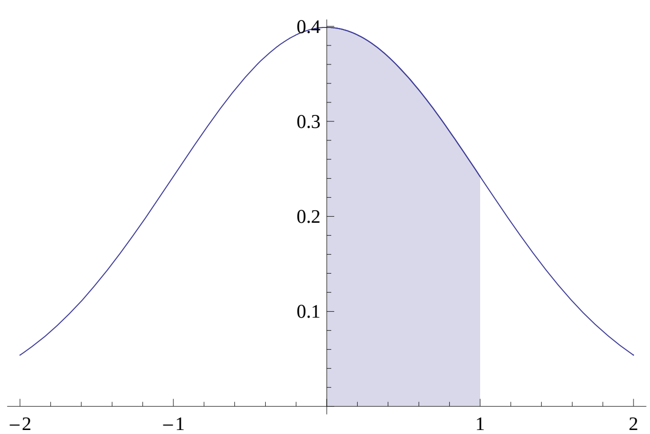
Problem 13

Problem. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the region bounded by

$$
\begin{aligned}
&y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \\
&y = 0, \\
&x = 0, \\
&x = 1
\end{aligned}
$$

about the y-axis.

Solution. This function is the standard normal curve, which is used extensively in probability and statistics.



The height is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. The volume is

$$\begin{aligned} V &= \int_0^1 2\pi x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \\ &= \frac{2\pi}{\sqrt{2\pi}} \int_0^1 x e^{-x^2/2}\,dx. \end{aligned}$$

Let $u = -\frac{x^2}{2}$ and $du = -x\,dx$. Then

$$\begin{aligned} V &= -\frac{2\pi}{\sqrt{2\pi}} \int_0^1 (-x) e^{-x^2/2}\,dx \\ &= -\sqrt{2\pi} \int_0^{-1/2} e^u\,du \\ &= -\sqrt{2\pi}\,\left[e^u\right]_0^{-1/2} \\ &= -\sqrt{2\pi}\left(\frac{1}{\sqrt{e}} - 1\right) \\ &= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right) \end{aligned}$$

Problem 17

Problem. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the region bounded by

$$\begin{aligned} y &= \frac{1}{x}, \\ x &= 1, \\ x &= 2 \end{aligned}$$

about the x-axis.

Solution. Because we are rotating about the x-axis, the radius is y (not x) and we will integrate with respect to y. Furthermore, we must express the boundaries (left and right) as functions of y (not x).

The boundaries are $x = \frac{1}{y}$, $x = 1$, and $x = 2$. We need to set this up as two separate integrals because the right boundary changes at $y = \frac{1}{2}$. From $y = 0$ to

$y = \frac{1}{2}$, the right boundary is $x = 2$. From $y = \frac{1}{2}$ to $y = 1$, the right boundary is $x = \frac{1}{y}$. The first integral is

$$\begin{aligned} V_1 &= \int_0^{1/2} 2\pi y(1)\,dy \\ &= 2\pi \int_0^{1/2} y\,dy \\ &= 2\pi \left[\frac{1}{2}y^2\right]_0^{1/2} \\ &= 2\pi \left(\frac{1}{2} \cdot 14\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

The second integral is

$$\begin{aligned} V &= \int_{1/2}^{1} 2\pi y \left(\frac{1}{y} - 1\right)\,dy \\ &= 2\pi \int_{1/2}^{1} (1-y)\,dy \\ &= 2\pi \left[y - \frac{1}{2}y^2\right]_{1/2}^{1} \\ &= 2\pi \left(\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right)\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

Thus, the volume of the solid is

$$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Problem 27

Problem. Decide whether it is more convenient to use the disk method of the shell method to find the volume of the solid of revolution bounded by

$$\begin{gathered} (y-2)^2 = 4 - x, \\ x = 0 \end{gathered}$$

about the x-axis.

Solution. The rotation is about the x-axis. Therefore, if we use the disk method, then we must integrate with respect to x, which means that we must express the upper and lower boundaries as functions of x. So we must solve the first equation for y (as a function of x):

$$\begin{aligned}(y-2)^2 &= 4-x,\\ y-2 &= \pm\sqrt{4-x},\\ y &= 2\pm\sqrt{4-x}.\end{aligned}$$

The two boundaries are $y = 2 - \sqrt{4-x}$ and $y = 2 + \sqrt{4-x}$. Yuck!

If we use the shell method, then we must integrate with respect to y (again, because we are rotating about the x-axis) and express the boundaries as functions of y. The boundaries would be $x = 0$ and $x = 4 - (y-2)^2$. Not bad. Not bad at all.

I would choose to use the shell method for this problem.

Problem 29

Problem. Use the disk method *or* the shell method to find the volume of the solid generated by revolving the region bounded by the graphs of the equations

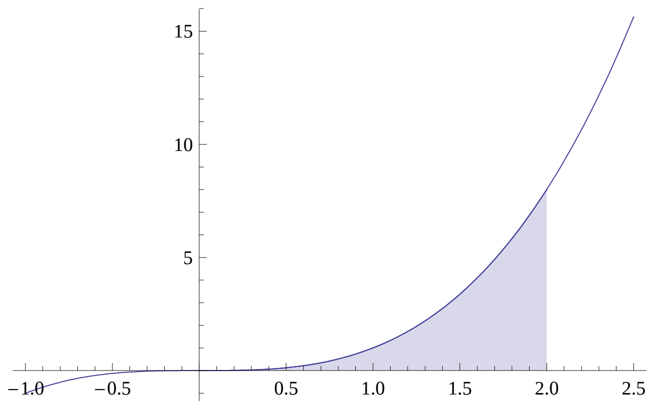
$$\begin{aligned}y &= x^3,\\ y &= 0,\\ x &= 2\end{aligned}$$

(a) the x-axis

(b) the y-axis

(c) the line $x = 4$

Solution. Here is the graph.



(a) We are rotating about the x-axis, so the disk method requires integration with respect to x and the shell method requires integration with respect to y. The one "complicated" function, $y = x^3$, is given as a function of x, so it ought to be easier to use the disk method.

$$\begin{aligned} V &= \int_0^2 \pi (x^3)^2 \, dx \\ &= \pi \int_0^2 x^6 \, dx \\ &= \pi \left[\frac{1}{7} x^7 \right]_0^2 \\ &= \frac{128\pi}{7}. \end{aligned}$$

(b) Now we are rotating about the y-axis. For the same reason as in part (a), it would be easier to integrate with respect to x. That will require that we use the shell method.

$$\begin{aligned} V &= \int_0^2 2\pi x (x^3) \, dx \\ &= 2\pi \int_0^2 x^4 \, dx \\ &= 2\pi \left[\frac{1}{5} x^5 \right]_0^2 \\ &= \frac{64\pi}{5}. \end{aligned}$$

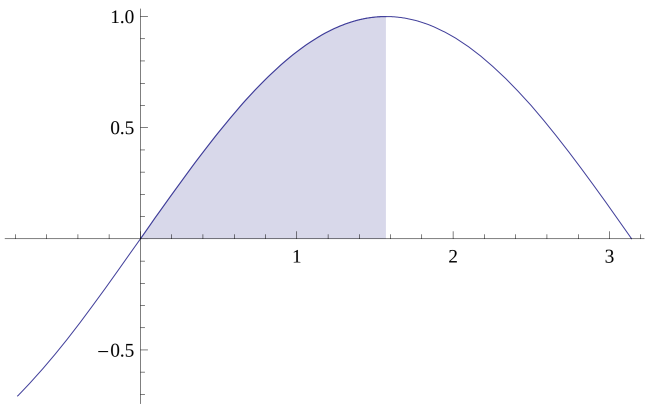
Problem 51

Problem. (a) Use differentiation to verify that

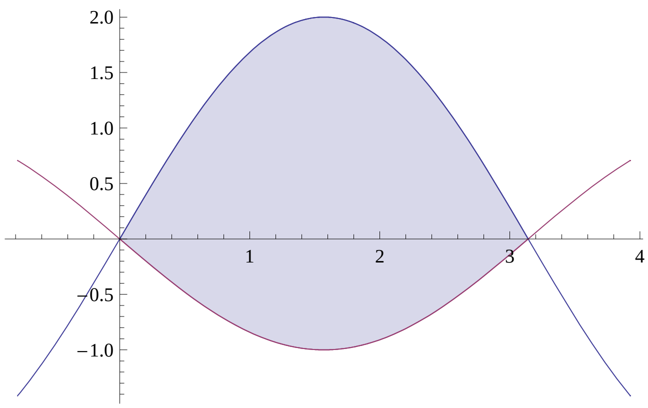
$$\int x \sin x \, dx = \sin x - x \cos x + C.$$

(b) Use the result of part (a) to find the volume of the solid generated by revolving the of the plane regions about the y-axis.

(i) $y = \sin x, y = 0, 0 \leq x \leq \frac{\pi}{2}$



(ii) $y = 2\sin x, y = -\sin x, 0 \leq x \leq \pi$



Solution. (a) Let $y = \sin x - x\cos x$. Then

$$\begin{aligned} y' &= \cos x - (1 \cdot \cos x + x(-\sin x)) \\ &= \cos x - \cos x + x \sin x \\ &= x \sin x. \end{aligned}$$

(b) (i) The volume is

$$\begin{aligned} V &= \int_0^{\pi/2} 2\pi x \sin x \; dx \\ &= 2\pi \left[\sin x - x\cos x\right]_0^{\pi/2} \\ &= 2\pi\left((1-0)-(0-0)\right) \\ &= 2\pi. \end{aligned}$$

(ii) The height of each shell is $2\sin x - (-\sin x) = 3\sin x$. The volume is

$$\begin{aligned} V &= \int_0^{\pi} 2\pi x (3\sin x) \; dx \\ &= 6\pi \int_0^{\pi} x\sin x \; dx \\ &= 6\pi \left[\sin x - x\cos x\right]_0^{\pi} \\ &= 2\pi\left((0-(-\pi))-(0-0)\right) \\ &= 2\pi^2. \end{aligned}$$

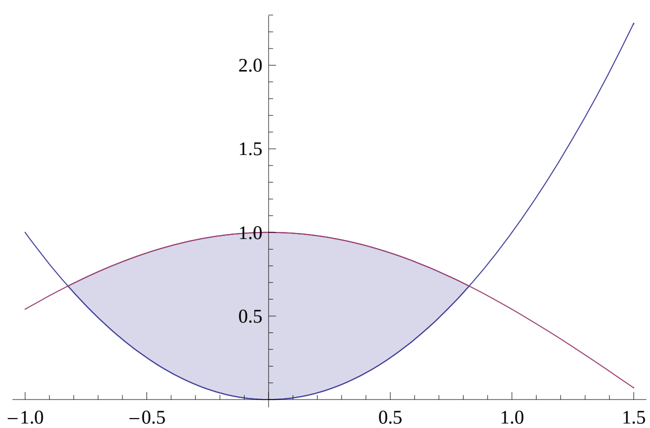
Problem 52

Problem. (a) Use differentiation to verify that

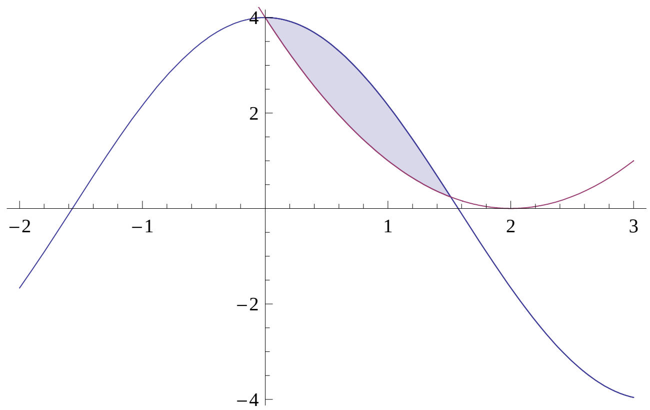
$$\int x\cos x \; dx = \cos x + x\sin x + C.$$

(b) Use the result of part (a) to find the volume of the solid generated by revolving the of the plane regions about the y-axis. (Begin by approximating the points of intersection.)

(i) $y = x^2, y = \cos x$



(ii) $y = 4\cos x, y = (x-2)^2$



Solution. (a) Let $y = \cos x + x \sin x$. Then

$$\begin{aligned} y' &= -\sin x + (1 \cdot \sin x + x\cos x) \\ &= -\sin x + \sin x + x\cos x \\ &= x\cos x. \end{aligned}$$

(b) (i) Using a numerical feature such as zero or intersect on the TI-83, we can approximate the points of intersection of $y = x^2$ and $y = \cos x$. The TI-83 reports that the intersection points occur at $x = -0.82413231$ and $x = 0.82413231$.

The height of a shell is $\cos x - x^2$, so the volume is

$$\begin{aligned} V &= \int_{-0.82413231}^{0.82413231} 2\pi x(\cos x - x^2)\, dx \\ &= 2\pi \int_{-0.82413231}^{0.82413231} (x\cos x - x^3)\, dx \\ &= 2\pi \left[\cos x + x\sin x - \frac{1}{3}x^3\right]_{-0.82413231}^{0.82413231} \\ &= 2\pi\,(1.470655097 - 1.097491248) \\ &= 0.7463276976\pi. \end{aligned}$$

(ii) Using a numerical feature such as zero or intersect on the TI-83, we can approximate the points of intersection of $y = x^2$ and $y = \cos x$. It is clear that the leftmost intersection point is at $x = 0$. The TI-83 reports that the rightmost intersection point occurs at $x = 1.5109741$.

The height of a shell is $4\cos x - (x-2)^2$, so the volume is

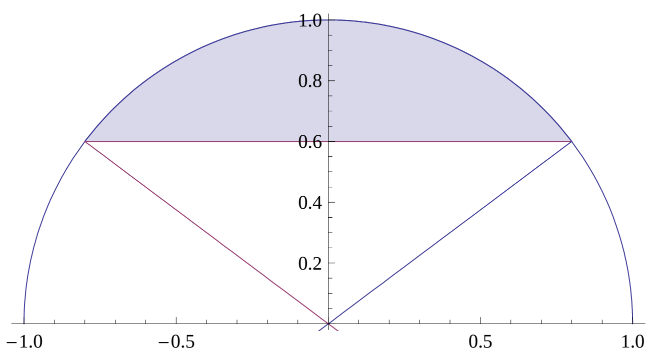
$$\begin{aligned}
V &= \int_0^{1.5109741} 2\pi x(4\cos x - (x-2)^2)\,dx \\
&= 2\pi \int_0^{1.5109741} (4x\cos x - x(x-2)^2)\,dx \\
&= 8\pi \int_0^{1.5109741} x\cos x\,dx + 2\pi \int_0^{1.5109741} (-x^3 + 4x^2 - 4x)\,dx \\
&= 8\pi\left[\cos x + x\sin x\right]_0^{1.5109741} + 2\pi\left[-\frac{1}{4}x^4 + \frac{4}{3}x^3 - 2x^2\right]_0^{1.5109741} \\
&= 8\pi(1.568057798 - 1) + 2\pi(-1.269665242) \\
&= 7.083792867\pi.
\end{aligned}$$

Problem 53

Problem. Let a sphere of radius r be cut by a plane, thereby forming a segment of height h. Show that the volume of this segment is

$$\frac{1}{3}\pi h^2(3r-h).$$

Solution. The problem is referring to the lopped-off part of the sphere, such as a polar cap. The following diagram shows a cross-section. We should rotate the right half of that region around the y-axis to get the desired volume.



The distance from the x-axis to the bottom of the shaded region is $r-h$ and the diagonal lines are radii (length r). Therefore, the extremities of the right half of the shaded region are from $x=0$ to $x=\sqrt{r^2-(r-h)^2}=\sqrt{2rh-h^2}$. The height of a shell is $\sqrt{r^2 - x^2} - (r - h)$. Now we can find the volume.

$$\begin{aligned} V &= \int_0^{\sqrt{2rh-h^2}} 2\pi x \left(\sqrt{r^2 - x^2} - (r - h)\right)\, dx \\ &= 2\pi \int_0^{\sqrt{2rh-h^2}} x\sqrt{r^2 - x^2}\, dx - 2\pi(r - h) \int_0^{\sqrt{2rh-h^2}} x\, dx. \end{aligned}$$

For the first integral, let $u = r^2 - x^2$ and $du = -2x\, dx$. Note that $u(0) = r^2$ and $u(\sqrt{2rh - h^2}) = (r - h)^2$. Then

$$\begin{aligned} 2\pi \int_0^{\sqrt{2rh-h^2}} x\sqrt{r^2 - x^2}\, dx &= -\pi \int_0^{\sqrt{2rh-h^2}} (-2x)\sqrt{r^2 - x^2}\, dx \\ &= -\pi \int_{r^2}^{(r-h)^2} \sqrt{u}\, du \\ &= -\pi \left[\frac{2}{3} u^{3/2}\right]_{r^2}^{(r-h)^2} \\ &= -\frac{2\pi}{3}\left((r - h)^3 - r^3\right). \end{aligned}$$

The second integral is

$$\begin{aligned} 2\pi(r - h) \int_0^{\sqrt{2rh-h^2}} x\, dx &= 2\pi(r - h)\left[\frac{1}{2}x^2\right]_0^{\sqrt{2rh-h^2}} \\ &= \pi(r - h)\left(2rh - h^2\right). \end{aligned}$$

Subtracting the second integral from the first integral gives us the volume.

$$\begin{aligned} V &= -\frac{2\pi}{3}\left((r - h)^3 - r^3\right) - \pi(r - h)\left(2rh - h^2\right) \\ &= \pi\left[-\frac{2}{3}\left(r^3 - 3r^2h + 3rh^2 - h^3 - r^3\right) - \left(2r^2h - 3rh^2 + h^3\right)\right] \\ &= \pi\left(2r^2h - 2rh^2 + \frac{2}{3}h^3 - 2r^2h + 3rh^2 - h^3\right) \\ &= \frac{1}{3}\pi\left(3rh^2 - h^3\right) \\ &= \frac{1}{3}\pi h^2(3r - h). \end{aligned}$$