Monday, October 5, 2015

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Problem 3

Problem. Find the arc length of the graph of the function $y = \frac{2}{3}(x^2+1)^{3/2}$ over the interval $[0, 1]$.

Solution. We find $y' = (x^2+1)^{1/2}(2x) = 2x\sqrt{x^2+1}$. Then

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+4x^2(x^2+1)}\\ &= \sqrt{1+4x^4+4x^2}\\ &= 2x^2+1.\end{aligned}$$

The arc length is

$$\begin{aligned}s &= \int_0^1 (2x^2+1)\,dx\\ &= \left[\frac{2}{3}x^3+x\right]_0^1\\ &= \frac{5}{3}.\end{aligned}$$

Problem 4

Problem. Find the arc length of the graph of the function $y = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $[1, 2]$.

Solution. We find $y' = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$. Then

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{1}{4}\left(x^2-\frac{1}{x^2}\right)^2}\\ &= \sqrt{1+\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4x^4}}\\ &= \sqrt{\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4x^4}}\\ &= \frac{1}{2}\left(x^2+\frac{1}{x^2}\right).\end{aligned}$$

The arc length is

$$\begin{aligned} s &= \int_1^2 \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\, dx \\ &= \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{x}\right]_1^2 \\ &= \frac{1}{2}\left(\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)\right) \\ &= \frac{17}{12}. \end{aligned}$$

Problem 9

Problem. Find the arc length of the graph of the function $y = \dfrac{x^5}{10} + \dfrac{1}{6x^3}$ over the interval $[2, 5]$.

Solution. We find $y' = \dfrac{x^4}{2} - \dfrac{1}{2x^4} = \dfrac{1}{2}\left(x^4 - \dfrac{1}{x^4}\right)$. Then

$$\begin{aligned} \sqrt{1 + (y')^2} &= \sqrt{1 + \frac{1}{4}\left(x^4 - \frac{1}{x^4}\right)^2} \\ &= \sqrt{1 + \frac{1}{4}x^8 - \frac{1}{2} + \frac{1}{4x^8}} \\ &= \sqrt{\frac{1}{4}x^8 + \frac{1}{2} + \frac{1}{4x^8}} \\ &= \frac{1}{2}\left(x^4 + \frac{1}{x^4}\right). \end{aligned}$$

The arc length is

$$\begin{aligned} s &= \int_2^5 \frac{1}{2}\left(x^4 + \frac{1}{x^4}\right)\, dx \\ &= \frac{1}{2}\left[\frac{1}{5}x^5 - \frac{1}{3x^3}\right]_2^5 \\ &= \frac{1}{2}\left(\left(625 - \frac{1}{375}\right) - \left(\frac{32}{5} - \frac{1}{24}\right)\right) \\ &= \frac{618639}{2000}. \end{aligned}$$

Problem 11

Problem. Find the arc length of the graph of the function $y = \ln \sin x$ over the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Solution. We find $y' = \dfrac{\cos x}{\sin x} = \cot x$. Then

$$\begin{aligned}
\sqrt{1 + (y')^2} &= \sqrt{1 + \cot^2 x} \\
&= \sqrt{\csc^2 x} \\
&= \csc x.
\end{aligned}$$

The arc length is

$$\begin{aligned}
s &= \int_{\pi/4}^{3\pi/4} \csc x \; dx \\
&= [-\ln(\csc x + \cot x)]_{\pi/4}^{3\pi/4} \\
&= -\ln\left(\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}\right) + \ln\left(\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right) \\
&= -\ln\left(\sqrt{2} - 1\right) + \ln\left(\sqrt{2} + 1\right) \\
&= \ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right) \\
&= \ln(3 + 2\sqrt{2}).
\end{aligned}$$

Problem 14

Problem. Find the arc length of the graph of the function $y = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right)$ over the interval $[\ln 2, \ln 3]$.

Solution. Rewrite y as $y = \ln(e^x + 1) - \ln(e^x - 1)$ and differentiate.

$$\begin{aligned}
y' &= \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} \\
&= \frac{e^x(e^x - 1) - e^x(e^x + 1)}{(e^x + 1)(e^x - 1)} \\
&= -\frac{2e^x}{(e^{2x} - 1)}.
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{1+(y')^2} &= \sqrt{1+\left(-\frac{2e^x}{(e^{2x}-1)}\right)^2} \\
&= \sqrt{\frac{(e^{2x}-1)^2}{(e^{2x}-1)^2}+\frac{4e^{2x}}{(e^{2x}-1)^2}} \\
&= \sqrt{\frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2}} \\
&= \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}} \\
&= \frac{e^{2x}+1}{e^{2x}-1} \\
&= \frac{e^x+e^{-x}}{e^x-e^{-x}}
\end{aligned}$$

Now we integrate to get the arc length.

$$\begin{aligned}
s &= \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx \\
&= \left[\ln\left|e^x-e^{-x}\right|\right]_{\ln 2}^{\ln 3} \\
&= \ln\left(e^{\ln 3}-e^{-\ln 3}\right)-\ln\left(e^{\ln 3}-e^{-\ln 3}\right) \\
&= \ln\left(3-\frac{1}{3}\right)-\ln\left(2-\frac{1}{2}\right) \\
&= \ln\frac{16}{9}.
\end{aligned}$$

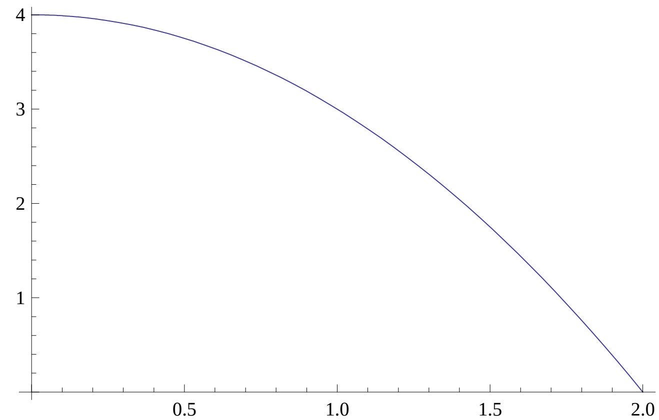
Problem 17

Problem. (a) Sketch the graph of the function $y = 4 - x^2$, $0 \leq x \leq 2$.

(b) Find a definite integral that represents the arc length over the interval.

(c) Use the integration capabilities of a graphing utility to approximate the arc length.

Solution. (a) The sketch of the graph:



(b) We have $y' = -2x$, so

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4x^2}.$$

The integral is

$$s = \int_0^2 \sqrt{1 + 4x^2}\, dx.$$

(c) Using the TI-83, I get $s = 4.64678$.

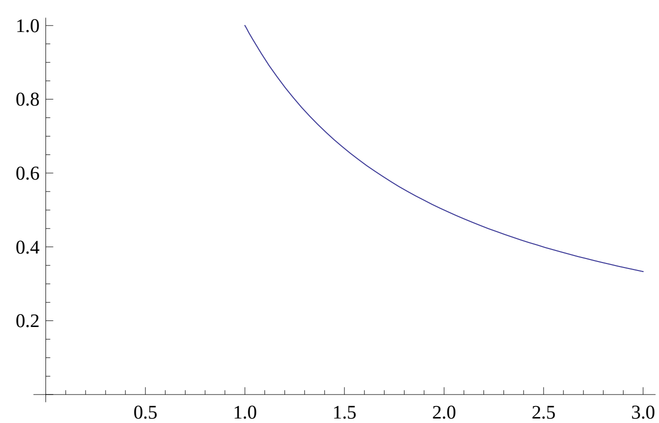
Problem 19

Problem. (a) Sketch the graph of the function $y = \dfrac{1}{x}$, $1 \le x \le 3$.

(b) Find a definite integral that represents the arc length over the interval.

(c) Use the integration capabilities of a graphing utility to approximate the arc length.

Solution. (a) The sketch of the graph:



(b) We have $y' = -\frac{1}{x^2}$, so

$$\sqrt{1+(y')^2} = \sqrt{1+\left(-\frac{1}{x^2}\right)^2}$$
$$= \sqrt{1+\frac{1}{x^4}}.$$

The integral is

$$s = \int_1^3 \sqrt{1+\frac{1}{x^4}}\,dx.$$

(c) Using the TI-83, I get $s = 2.14662$.

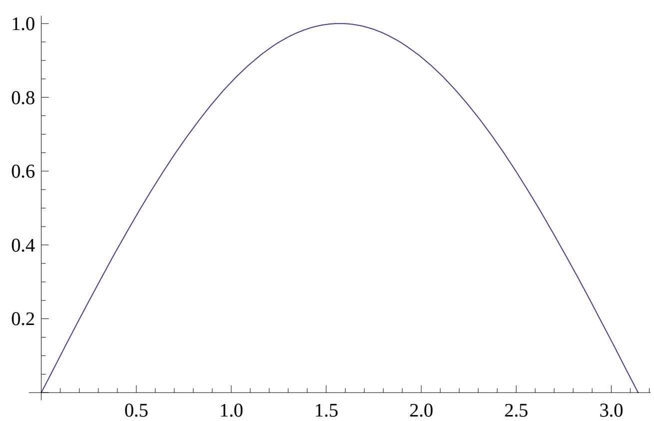
Problem 21

Problem. (a) Sketch the graph of the function $y = \sin x$, $0 \le x \le \pi$.

(b) Find a definite integral that represents the arc length over the interval.

(c) Use the integration capabilities of a graphing utility to approximate the arc length.

Solution. (a) The sketch of the graph:



(b) We have $y' = \cos x$, so

$$\sqrt{1+(y')^2} = \sqrt{1+\cos^2 x}.$$

The integral is

$$s = \int_0^\pi \sqrt{1+\cos^2 x}\,dx.$$

(c) Using the TI-83, I get $s = 3.82020$.

Problem 23

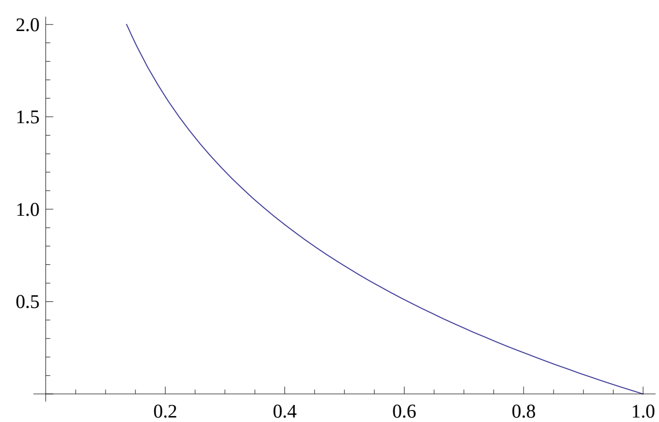
Problem. (a) Sketch the graph of the function $x = e^{-y}$, $0 \le y \le 2$.

(b) Find a definite integral that represents the arc length over the interval.

(c) Use the integration capabilities of a graphing utility to approximate the arc length.

Solution. In this problem, the roles of x and y are reversed. You may work it with x and y reversed, or you may rewrite the function as $y = -\ln x$ and the interval as $e^{-2} \le x \le 1$. Either way, you get the same answer.

(a) The function is the same as the function $y = -\ln x$. The sketch of the graph:



(b) We have $x' = -e^{-y}$, so

$$\begin{aligned}\sqrt{1 + (x')^2} &= \sqrt{1 + (-e^{-y})^2} \\ &= \sqrt{1 + e^{-2y}}.\end{aligned}$$

The integral is

$$s = \int_0^2 \sqrt{1 + e^{-2y}}\, dy.$$

(c) Using the TI-83, I get $s = 2.22142$.

Problem 34

Problem. Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$.

Solution. Note: this is an *astroid*, not an asteroid.

Solve for y and get

$$y = \left(4 - x^{2/3}\right)^{3/2}.$$

Now differentiate.

$$\begin{aligned} y' &= \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) \\ &= -x^{-1/3}\left(4 - x^{2/3}\right)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} \sqrt{1 + (y')^2} &= \sqrt{1 + \left(-x^{-1/3}\left(4 - x^{2/3}\right)^{1/2}\right)^2} \\ &= \sqrt{1 + x^{-2/3}\left(4 - x^{2/3}\right)} \\ &= \sqrt{1 + 4x^{-2/3} - 1} \\ &= \sqrt{4x^{-2/3}} \\ &= 2x^{-1/3}. \end{aligned}$$

Now integrate to find the arc length.

$$\begin{aligned} s &= \int_0^8 2x^{-1/3}\, dx \\ &= 2\left[\frac{3}{2}x^{2/3}\right]_0^8 \\ &= 2 \cdot 6 \\ &= 12. \end{aligned}$$

So the distance around all four sides is $4 \cdot 12 = 48$.