Tuesday, October 13, 2015

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Problem 9

Problem. Find the center of mass in the system of point masses

m_i	5	1	3
(x_i, y_i)	$(2,2)$	$(-3,1)$	$(1,-4)$

Solution.

$$\begin{aligned}\overline{x} &= \frac{(5\cdot 2) + (1\cdot(-3)) + (3\cdot 1)}{5+1+3} = \frac{10}{9},\\ \overline{y} &= \frac{(5\cdot 2) + (1\cdot 1) + (3\cdot(-4))}{5+1+3} = -\frac{1}{9}.\end{aligned}$$

Problem 13

Problem. Find M_x, M_y, and $(\overline{x}, \overline{y})$ for the lamina of uniform density ρ bounded by

$$y = \frac{1}{2}x, \quad y = 0, \quad x = 2$$

Solution.

$$\begin{aligned}M_y &= \int_0^2 \frac{1}{2}x \cdot x\, dx\\ &= \frac{1}{2}\int_0^2 x^2\, dx\\ &= \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2\\ &= \frac{1}{6}(8)\\ &= \frac{4}{3}.\end{aligned}$$

$$
\begin{aligned}
M_x &= \int_0^1 (2-2y)\,y\;dy \\
&= \int_0^1 \left(2x-2y^2\right)\;dy \\
&= \left[y^2 - \frac{2}{3}y^3\right]_0^1 \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Area} &= \int_0^2 \frac{1}{2}x \cdot x\;dx \\
&= \left[\frac{1}{4}x^2\right]_0^2 \\
&= \frac{1}{4}(4) \\
&= 1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\overline{x} &= \frac{4}{3}, \\
\overline{y} &= \frac{1}{3}.
\end{aligned}
$$

Problem 15

Problem. Find M_x, M_y, and $(\overline{x}, \overline{y})$ for the lamina of uniform density ρ bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

Solution.

Problem 17

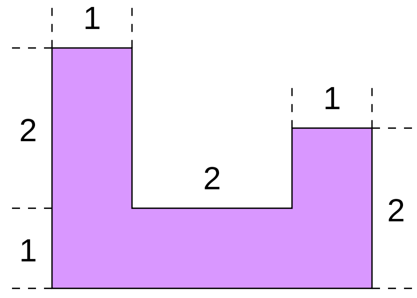
Problem. Find M_x, M_y, and $(\overline{x}, \overline{y})$ for the lamina of uniform density ρ bounded by

$$y = x^2, \quad y = x^3$$

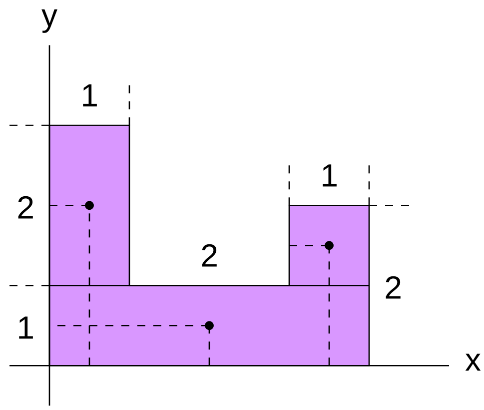
Solution.

Problem 32

Problem. Introduce an appropriate coordinate system and find the coordinates of the center of mass of the planar lamina.



Solution. Let the x-axis be the bottom of the figure and the y-axis the left edge. Partition the figure into three rectangles, as shown in the following diagram.



The centroids of the rectangles are their geometric centers, which are $(0.5, 2)$, $(2, 0.5)$, and $(3.5, 1.5)$, respectively. Their areas (masses) are 2, 4, and 1, respectively. Now we have a situation identical to Exercise 9.

m_i	2	4	1
(x_i, y_i)	$(0.5, 2)$	$(2, 0.5)$	$(3.5, 1.5)$

so we solve it the same way.

$$\begin{aligned}\overline{x} &= \frac{(2 \cdot 0.5) + (4 \cdot 2) + (1 \cdot 3.5)}{2 + 4 + 1} \\ &= \frac{12.5}{7} \\ &= \frac{25}{14}\end{aligned}$$

and

$$\begin{aligned}\overline{y} &= \frac{(2 \cdot 2) + (4 \cdot 0.5) + (1 \cdot 1.5)}{2 + 4 + 1} \\ &= \frac{7.5}{7} \\ &= \frac{15}{14}\end{aligned}$$

The centroid is $(\frac{25}{14}, \frac{15}{14})$.

Problem 37

Problem. Use the Theorem of Pappus to find the volume of the torus formed by revolving the circle

$$(x - 5)^2 + y^2 = 16$$

about the y-axis.

Solution. The centroid of the circle is the center of the circle, which is $(5, 0)$. The radius of the circle is $\sqrt{16} = 4$, so the area of the circle is $A = \pi(4)^2 = 16\pi$. Because we are revolving about the y-axis, the radius of the torus is 5. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = 2\pi r A = 2\pi(5)(16\pi) = 160\pi^2.$$

Problem 39

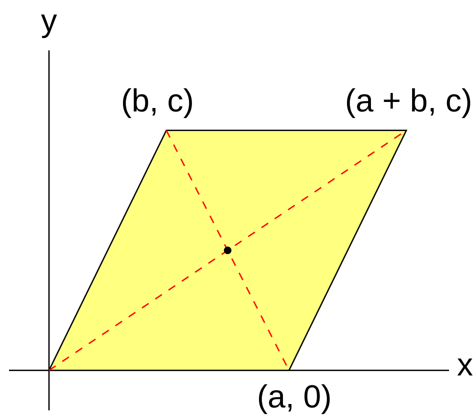
Problem. Use the Theorem of Pappus to find the volume of the solid formed by revolving the region bounded by the graphs of $y = x$, $y = 4$, and $x = 0$ about the x-axis.

Solution. We need to find the centroid of the triangle. The top edge is the "base" and it has length 4, so $\overline{x} = \frac{4}{3}$. The vertical edge is the height and it also has length 4, so $\overline{y} = \frac{8}{3}$ (because we measure 1/3 the way from the base, i.e., down from $y = 4$). Actually, all we need is $\overline{y}$ because that is the radius when revolving about the x-axis. The area of the triangle is $A = \frac{1}{2} \cdot 4 \cdot 4 = 8$. Therefore, by the Theorem of Pappus, the volume is

$$V = 2\pi r A = 2\pi \left(\frac{8}{3}\right)(8) = \frac{128\pi}{3}.$$

Problem 45

Problem. Find and verify the centroid of the parallelogram.



Solution. The non-calculus way to work it is to divide the parallelogram into two triangles along the line from (b, c) to $(a, 0)$.

Consider the left triangle and its median from $(0, 0)$ to its opposite side. The centroid is $\frac{2}{3}$ the way along that median from $(0, 0)$ to $\left(\frac{a+b}{2}, \frac{c}{2}\right)$, so the centroid is at $\left(\frac{a+b}{3}, \frac{c}{3}\right)$.

Now consider the right-hand triangle and its median from $(a + b, c)$ to $\left(\frac{a+b}{2}, \frac{c}{2}\right)$. Its centroid is $\frac{1}{3}$ the way from $\left(\frac{a+b}{2}, \frac{c}{2}\right)$ to $(a + b, c)$, which is $\left(\frac{2(a+b)}{3}, \frac{2c}{3}\right)$. Because the two triangles have equal area, the centroid of their combination is the midpoint of the line connecting their centroids, namely $\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

To work it using calculus, we have to find functions that describe the boundary of the parallelogram and then use integration. The boundaries are

$$\begin{aligned} y &= \frac{c}{b}x, \\ y &= c, \\ y &= \frac{c}{b}x - \frac{ac}{b}. \end{aligned}$$

With great patience, we find that

$$\begin{aligned} \int_0^b \frac{c}{b}x^2\,dx &= \frac{b^2c}{3}, \\ \int_b^a cx\,dx &= \frac{a^2c}{2} - \frac{b^2c}{2}, \\ \int_a^{a+b} \left(c - \left(\frac{c}{b}x - \frac{ac}{b}\right)\right) x\,dx &= \frac{abc}{2} + \frac{b^2c}{6}. \end{aligned}$$

Adding these up, we get

$$\begin{aligned} M_y &= \frac{b^2c}{3} + \frac{a^2c}{2} - \frac{b^2c}{2} + \frac{abc}{2} + \frac{b^2c}{6} \\ &= \frac{ac(a+b)}{2}. \end{aligned}$$

The area of the parallelogram is ac, so

$$\begin{aligned} \overline{x} &= \frac{\left(\frac{ac(a+b)}{2}\right)}{ac} \\ &= \frac{a+b}{2}. \end{aligned}$$

In the y-direction, the left and right boundaries are

$$\begin{aligned} x &= \frac{b}{c}y, \\ x &= \frac{b}{c}y + a. \end{aligned}$$

Then

$$\begin{aligned} M_x &= \int_0^c \left(\left(\frac{b}{c}y + a\right) - \frac{b}{c}y\right) y\,dy \\ &= \int_0^c ay\,dy \\ &= \left[\frac{a}{2}y^2\right]_0^c \\ &= \frac{ac^2}{2}. \end{aligned}$$

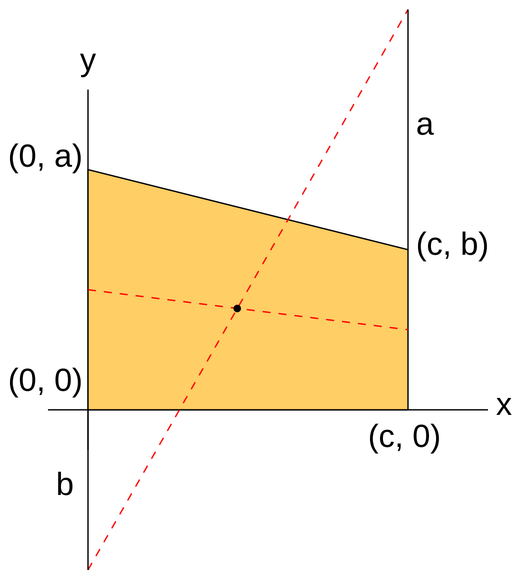
Then

$$\overline{y} = \frac{\left(\frac{ac^2}{2}\right)}{ac}$$
$$= \frac{c}{2}.$$

The centroid is $\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

Problem 46

Problem. Find and verify the centroid of the trapezoid.



Solution. To find the point indicated in the diagram, we could write the equations of the two lines and then find their point of intersection. The lines are

$$y = -\left(\frac{b-a}{2c}\right)x + a,$$
$$y = \left(\frac{a+2b}{c}\right)x - b.$$

Solving for the intersection, we get

$$x = \frac{(a+2b)c}{3(a+b)},$$
$$y = \frac{a(a+3b)}{2b}.$$

Using integration, the upper boundary is $y = \left(\dfrac{b-a}{c}\right)x + a$, so

$$\begin{aligned}
M_y &= \int_0^c \left(\frac{b-a}{c}x + a\right) x \; dx \\
&= \int_0^c \left(\frac{b-a}{c}x^2 + ax\right) \; dx \\
&= \left[\frac{b-a}{3c}x^3 + \frac{a}{2}x^2\right]_0^c \\
&= \frac{b-a}{3c}c^3 + \frac{a}{2}c^2 \\
&= \frac{(b-a)c^2}{3} + \frac{ac^2}{2} \\
&= \frac{(a+2b)c^2}{6}.
\end{aligned}$$

The area of the trapezoid is $\frac{(a+b)c}{2}$, so

$$\begin{aligned}
\overline{x} &= \frac{\left(\frac{(a+2b)c^2}{6}\right)}{\left(\frac{(a+b)c}{2}\right)} \\
&= \frac{(a+2b)c^2}{6} \cdot \frac{2}{(a+b)c} \\
&= \frac{(a+2b)c}{3(a+b)}.
\end{aligned}$$