Wednesday, October 21, 2015

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Problem 1

Problem. Find the indefinite integral $\int \cos^5 x \sin x \, dx$.

Solution. $\sin x$ and $\cos x$ are both to odd powers, but it will be much easier to use $u = \cos x$, $du = -\sin x \, dx$. In fact, this problem is nothing but a straightforward substitution problem.

$$\begin{aligned} \int \cos^5 x \sin x \, dx &= -\int \cos^5 x(-\sin x) \, dx \\ &= -\int u^5 \, du \\ &= -\frac{1}{6}u^6 + C \\ &= -\frac{1}{6}\cos^6 x + C. \end{aligned}$$

Problem 2

Problem. Find the indefinite integral $\int \cos^3 x \sin^4 x \, dx$.

Solution. $\cos x$ is to an odd power, so replace $\cos^2 x$ with $1 - \sin^2 x$ and then use the substitution $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int \cos^3 x \sin^4 x \, dx &= \int \cos x(1 - \sin^2 x) \sin^4 x \, dx \\ &= \int (1 - u^2)u^4 \, du \\ &= \int (u^4 - u^6) \, du \\ &= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C \\ &= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C. \end{aligned}$$

Problem 3

Problem. Find the indefinite integral $\int \sin^7 2x \cos 2x \, dx$.

Solution. This is similar to Exercise 1. $\cos 2x$ is to the first power. So it is just a simple substitution problem. Let $u = \sin 2x$ and $du = 2\cos 2x\, dx$.

$$\begin{aligned}\int \sin^7 2x \cos 2x\, dx &= \frac{1}{2}\int \sin^7 2x(2\cos 2x)\, dx \\ &= \frac{1}{2}\int u^7\, du \\ &= \frac{1}{16}u^8 + C \\ &= \frac{1}{16}\sin^8 2x + C.\end{aligned}$$

Problem 4

Problem. Find the indefinite integral $\int \sin^3 3x\, dx$.

Solution. $\sin 3x$ is to an odd power and $\cos 3x$ to an even power (the 0 power). So we must use $\sin^2 3x = 1 - \cos^2 3x$ and let $u = \cos 3x$, $du = -3\sin 3x\, dx$.

$$\begin{aligned}\int \sin^3 3x\, dx &= \int (1-\cos^2 3x)\sin 3x\, dx \\ &= -\frac{1}{3}\int (1-\cos^2 3x)(-3\sin 3x)\, dx \\ &= -\frac{1}{3}\int (1-u^2)\, du \\ &= -\frac{1}{3}\left(u - \frac{1}{3}u^3\right) + C \\ &= -\frac{1}{3}u + \frac{1}{9}u^3 + C \\ &= -\frac{1}{3}\cos 3x + \frac{1}{9}\cos^3 3x + C.\end{aligned}$$

Problem 5

Problem. Find the indefinite integral $\int \sin^3 x \cos^2 x\, dx$.

Solution. $\sin x$ is to an odd power and $\cos x$ to an even power. So we must use

$\sin^2 x = 1 - \cos^2 x$ and let $u = \cos x$, $du = -\sin x\ dx$.

$$\begin{aligned}
\int \sin^3 x \cos^2 x\ dx &= -\int (-\sin x)(1 - \cos^2 x) \cos^2 x\ dx \\
&= -\int (1 - u^2)u^2\ dx \\
&= -\int (u^2 - u^4)\ dx \\
&= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\
&= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.
\end{aligned}$$

Problem 9

Problem. Find the indefinite integral $\int \cos^2 3x\ dx$.

Solution. $\cos 3x$ and $\sin 3x$ are both to even powers, so we must use the identity $\cos^2 3x = \frac{1}{2}(1 + \cos 6x)$.

$$\begin{aligned}
\int \cos^2 3x\ dx &= \int \frac{1}{2}(1 + \cos 6x)\ dx \\
&= \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C \\
&= \frac{1}{2}x + \frac{1}{12}\sin 6x + C.
\end{aligned}$$

Problem 10

Problem. Find the indefinite integral $\int \sin^4 6\theta\ d\theta$.

Solution. $\sin 6\theta$ and $\cos 6\theta$ are both to even powers, so we must begin by using the identity $\sin^2 6\theta = \frac{1}{2}(1 - \cos 12\theta)$.

$$\begin{aligned}
\int \sin^4 6\theta\ d\theta &= \int \left(\frac{1}{2}(1 - \cos 12\theta)\right)^2\ d\theta \\
&= \frac{1}{4}\int \left(1 - 2\cos 12\theta + \cos^2 12\theta\right)\ d\theta \\
&= \frac{1}{4}\int\ d\theta - \frac{1}{2}\int \cos 12\theta\ d\theta + \frac{1}{4}\int \cos^2 12\theta\ d\theta
\end{aligned}$$

Problem 11

Problem. Find the indefinite integral $\int x \sin^2 x \, dx$.

Solution. We should begin by using integration by parts. Let $u = x$ and $dv = \sin^2 x \, dx$. Then $du = dx$ and $v = \int \sin^2 x \, dx$. Now, to find v, we need the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$$\begin{aligned} v &= \int \sin^2 x \, dx \\ &= \int \frac{1}{2}(1 - \cos 2x) \, dx \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x. \end{aligned}$$

Now we can finish the integration by parts.

$$\begin{aligned} \int x \sin^2 x \, dx &= x\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) - \int \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) \, dx \\ &= \frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8}\cos 2x. \qquad = \end{aligned}$$

Problem 12

Problem. Find the indefinite integral $\int x^2 \sin^2 x \, dx$.

Solution. This is similar to Exercise 11, except that we have x^2 instead of just x. Use integration by parts with $u = x^2$ and $dv = \sin^2 x \, dx$. Then $du = 2x \, dx$ and $v = \frac{1}{2}x - \frac{1}{4}\sin 2x$ (worked out in Exercise 11). So we have

$$\begin{aligned} \int x^2 \sin^2 x \, dx &= x^2\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) - \int \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)(2x) \, dx \\ &= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \int \left(x^2 - \frac{1}{2}x \sin 2x\right) \, dx \\ &= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x \sin 2x \, dx \\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2}\int x \sin 2x \, dx. \end{aligned}$$

We need integration by parts again to do the last integral. Let $u = x$ and $dv = \sin 2x\, dx$. Then $du = dx$ and $v = -\frac{1}{2}\cos 2x$. Now we complete the problem.

$$\begin{aligned}\int x^2 \sin^2 x\, dx &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\, dx\right)\\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x.\end{aligned}$$

Problem 13

Problem. Use Wallis's formula to evaluate the integral $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx$.

Solution. The exponent is odd, so we use the first form.

$$\int_0^{\pi/2} \cos^7 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = \frac{5}{16}.$$

Problem 15

Problem. Use Wallis's formula to evaluate the integral $\displaystyle\int_0^{\pi/2} \cos^{10} x\, dx$.

Solution. The exponent is even, so we use the second form.

$$\int_0^{\pi/2} \cos^{10} x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right) = \frac{63\pi}{512}.$$

Problem 18

Problem. Use Wallis's formula to evaluate the integral $\displaystyle\int_0^{\pi/2} \sin^8 x\, dx$.

Solution. The exponent is even, so we use the second form.

$$\int_0^{\pi/2} \sin^8 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right) = \frac{35\pi}{256}.$$