Wednesday, November 18, 2015

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Problem 1

Problem. Confirm that the Integral Test can be applied to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$. Then use the Integral Test to determine the convergence or divergence of the series.

Solution. Let $f(x) = \frac{1}{x+3}$. On the interval $[1, \infty)$, $f(x)$ is positive, continuous, and decreasing (all obvious), so the Integral Test applies.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x+3}\, dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x+3}\, dx \\
&= \lim_{t\to\infty} \left[\ln|x+3|\right]_1^t \\
&= \lim_{t\to\infty} \left(\ln(t+3) - \ln 4\right) \\
&= \infty.
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges.

Problem 4

Problem. Confirm that the Integral Test can be applied to the series $\displaystyle\sum_{n=1}^{\infty} 3^{-n}$. Then use the Integral Test to determine the convergence or divergence of the series.

Solution. Let $f(x) = 3^{-x}$. On the interval $[1, \infty)$, $f(x)$ is positive, continuous, and decreasing (all obvious), so the Integral Test applies.

$$\begin{aligned}
\int_1^{\infty} 3^{-x}\, dx &= \lim_{t\to\infty} \int_1^t 3^{-x}\, dx \\
&= \lim_{t\to\infty} \left[-\frac{3^{-x}}{\ln 3}\right]_1^t \\
&= \lim_{t\to\infty} \left(-\frac{3^{-t}}{\ln 3} + \frac{3^{-1}}{\ln 3}\right) \\
&= \frac{1}{3\ln 3}.
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} 3^{-n}$ converges.

Problem 6

Problem. Confirm that the Integral Test can be applied to the series $\sum_{n=1}^{\infty} ne^{-n/2}$. Then use the Integral Test to determine the convergence or divergence of the series.

Solution. Let $f(x) = xe^{-x/2}$. Then $f(x)$ is positive and continuous on $[1, \infty)$, but it is not clear that it is decreasing. We need to show that.

$$\begin{aligned} f'(x) &= e^{-x/2} + xe^{-x/2} \cdot \left(-\frac{1}{2}\right) \\ &= e^{-x/2}\left(1 - \frac{x}{2}\right). \end{aligned}$$

From that it is clear that $f(x)$ is decreasing on $[2, \infty)$, which is good enough, so the Integral Test applies.

$$\int_1^{\infty} xe^{-x/2}\, dx = \lim_{t\to\infty} \int_1^t xe^{-x/2}\, dx.$$

Now we need integration by parts. Let $u = x$ and $dv = e^{-x/2}\, dx$. Then $du = dx$ and $v = -2e^{-x/2}$.

$$\begin{aligned} \lim_{t\to\infty} \int_1^t xe^{-x/2}\, dx &= \lim_{t\to\infty} \left(\left[-2xe^{-x/2}\right]_1^t + 2\int_1^t e^{-x/2}\, dx \right) \\ &= \lim_{t\to\infty} \left(\left(-2te^{-t/2} + 2e^{-1/2}\right) - 4\left[e^{-x/2}\right]_1^t \right) \\ &= \lim_{t\to\infty} \left(\left(-2te^{-t/2} + 2e^{-1/2}\right) - 4\left(e^{-t/2} - e^{-1/2}\right) \right). \end{aligned}$$

The term $e^{-t/2}$ goes to 0. We need L'Hôpital's Rule to evaluate $\lim_{t\to\infty} (-2te^{-t/2})$.

$$\begin{aligned} \lim_{t\to\infty} -2te^{-t/2} &= \lim_{t\to\infty} \frac{-2t}{e^{t/2}} \\ &= \lim_{t\to\infty} \frac{-2}{\frac{1}{2}e^{t/2}} \\ &= 0. \end{aligned}$$

Therefore, $\int_1^{\infty} xe^{-x/2}\, dx = 6e^{-1/2}$ and therefore, $\sum_{n=1}^{\infty} ne^{-n/2}$ converges.

Problem 7

Problem. Confirm that the Integral Test can be applied to the series

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \cdots .$$

Then use the Integral Test to determine the convergence or divergence of the series.

Solution. The denominators appear to be terms in the sequence $n^2 + 1$. So the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Let $f(x) = \dfrac{1}{x^2+1}$. It is continuous, decreasing, and positive on $[1, \infty)$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx \\
&= \lim_{t\to\infty} [\arctan x]_1^t \\
&= \lim_{t\to\infty} (\arctan t - \arctan 1) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Problem 11

Problem. Confirm that the Integral Test can be applied to the series

$$\frac{1}{\sqrt{1}(\sqrt{1}+1)} + \frac{1}{\sqrt{2}(\sqrt{2}+1)} + \frac{1}{\sqrt{3}(\sqrt{3}+1)} + \cdots + \frac{1}{\sqrt{n}(\sqrt{n}+1)} + \cdots .$$

Then use the Integral Test to determine the convergence or divergence of the series.

Solution. Let $f(x) = \dfrac{1}{\sqrt{x}(\sqrt{x}+1)}$. Clearly, $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$. To find the integral $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$, we need to make the substitution $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\begin{aligned}
\int_1^\infty \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= 2\int_1^\infty \frac{1}{u+1}\,du \\
&= 2\lim_{t\to\infty}\int_1^t \frac{1}{u+1}\,du \\
&= 2\lim_{t\to\infty}\left[\ln|u+1|\right]_1^t \\
&= 2\lim_{t\to\infty}\left(\ln|t+1| - \ln 2\right) \\
&= \infty.
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ diverges.

Problem 15

Problem. Confirm that the Integral Test can be applied to the series $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2}$. Then use the Integral Test to determine the convergence or divergence of the series.

Solution. Let $f(x) = \frac{\ln x}{x^2}$. Clearly, $f(x)$ is positive and continuous on $[1,\infty)$. We need to show that it is decreasing.

$$\begin{aligned}
f'(x) &= \frac{x^2\cdot\frac{1}{x} - 2x\cdot\ln x}{x^4} \\
&= \frac{1-2\ln x}{x^3}.
\end{aligned}$$

This is negative when $2\ln x > 1$, which is when $x > e^{1/2}$, which is good enough. So the Integral Test applies.

To integrate $\dfrac{\ln x}{x^2}$, we need to use integration by parts. Let $u = \ln x$ and $dv = x^{-2}\,dx$. Then $du = x^{-1}\,dx$ and $v = -x^{-1}$.

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{t\to\infty}\left(\left[-\frac{\ln x}{x}\right]_1^t - \int_1^t x^{-2}\,dx\right) \\
&= \lim_{t\to\infty}\left(\left(-\frac{\ln t}{t}\right) - \left[-x^{-1}\right]_1^t\right) \\
&= \lim_{t\to\infty}\left(\left(-\frac{\ln t}{t}\right) - \left(-\frac{1}{t}+1\right)\right)
\end{aligned}$$

L'Hôpital's Rule shows that $\lim_{t\to\infty} \frac{\ln t}{t} = 0$, so $\int_1^\infty \frac{\ln x}{x^2}\,dx = 1$.

Therefore, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

Problem 24

Problem. Use the Integral Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} n^k e^{-n}$, where k is a positive integer.

Solution. Refer to Exercise 103 on page 575 (homework from 11/11/15). That exercise established that

$$\int_0^\infty x^{n-1}e^{-x}\,dx = n!.$$

It follows that the integral $\int_1^\infty x^{n-1}e^{-x}\,dx$ is finite and therefore converges. Therefore, $\sum_{n=1}^{\infty} n^k e^{-n}$ converges.

Problem 30

Problem. Use the Integral Test to determine the convergence or divergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

Solution. Let $f(x) = \frac{1}{x^{1/2}}$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^{1/2}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^{1/2}}\,dx \\
&= \lim_{t\to\infty} \left[2x^{1/2}\right]_1^t \\
&= \lim_{t\to\infty} \left(2\sqrt{t} - 2\right) \\
&= \infty.
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges.

Problem 33

Problem. Use Theorem 9.11 to determine the convergence or divergence of the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$.

Solution. The exponent is $p = \frac{1}{5} < 1$, therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ diverges.

Problem 34

Problem. Use Theorem 9.11 to determine the convergence or divergence of the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$.

Solution. The exponent is $p = \frac{5}{3} > 1$, therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$ converges. The 3 in the numerator does not matter.

Problem 55

Problem. Use the result of Exercise 53 to approximate the sum of the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ using five terms. Include an estimate of the maximum error for your approximation.

Solution. The sum of the first 5 terms is

$$\begin{aligned}\sum_{n=1}^{5} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \\ &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \\ &= \frac{5269}{3600} \\ &= 1.4636111\ldots.\end{aligned}$$

According to Exercise 53, the remainder R_5 is no greater than

$$\begin{aligned}\int_5^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_5^t \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_5^t\\ &= \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{5}\right)\\ &= \frac{1}{5}\\ &= 0.2.\end{aligned}$$

Thus,

$$1.4636111\ldots \leq \sum_{n=1}^{\infty}\frac{1}{n^2} \leq 1.6636111\ldots.$$

Problem 56

Problem. Use the result of Exercise 53 to approximate the sum of the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^5}$ using six terms. Include an estimate of the maximum error for your approximation.

Solution. The sum of the first 6 terms is

$$\begin{aligned}\sum_{n=1}^{6}\frac{1}{n^5} &= \frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}\\ &= 1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\frac{1}{7776}\\ &= 1.03679039\ldots.\end{aligned}$$

According to Exercise 53, the remainder R_6 is no greater than

$$\begin{aligned}\int_6^\infty \frac{1}{x^5}\,dx &= \lim_{t\to\infty}\int_6^t \frac{1}{x^5}\,dx\\ &= \lim_{t\to\infty}\left[-\frac{1}{4x^4}\right]_6^t\\ &= \lim_{t\to\infty}\left(-\frac{1}{4t^4}+\frac{1}{4\cdot 6^4}\right)\\ &= \frac{1}{5184}\\ &= 0.1929012346\ldots.\end{aligned}$$

Thus,

$$1.03679039\ldots \le \sum_{n=1}^{\infty}\frac{1}{n^5} \le 1.036983291\ldots.$$