Monday, November 23, 2015

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Problem 13

Problem. Use the Limit Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Solution. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{n}$ because, for large n, $n^2+1 \approx n^2$, so $\frac{n}{n^2+1} \approx \frac{n}{n^2} = \frac{1}{n}$.

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n^2}{n^2+1} \\ &= 1. \end{aligned}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Therefore, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

Problem 14

Problem. Use the Limit Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$.

Solution. This resembles 5 times the geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$.

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{5}{4^n+1}}{\frac{1}{4^n}} &= \lim_{n\to\infty} \frac{5 \cdot 4^n}{4^n+1} \\ &= \lim_{n\to\infty} \frac{5}{1+\frac{1}{4^n}} \\ &= 5. \end{aligned}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges (geometric, $r = \frac{1}{4}$). Therefore, $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$ converges.

Problem 15

Problem. Use the Limit Comparison Test to determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

Solution. Because $\sqrt{n^2+1} \approx n$, we will compare this series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} \\
&= 1.
\end{aligned}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

Problem 17

Problem. Use the Limit Comparison Test to determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$.

Solution. For large n, $2n^2-1 \approx 2n^2$ and $3n^5+2n+1 \approx 3n^5$. Therefore, $\displaystyle\frac{2n^2-1}{3n^5+2n+1} \approx \frac{2n^2}{3n^5} = \frac{2}{3n^3}$. We will compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{2n^2-1}{3n^5+2n+1}}{\frac{1}{n^3}} &= \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1} \\
&= \lim_{n\to\infty} \frac{2-\frac{1}{n^2}}{3+\frac{2}{n^4}+\frac{1}{n^5}} \\
&= \frac{2}{3}.
\end{aligned}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$ converges.

Problem 19

Problem. Use the Limit Comparison Test to determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$.

Solution. Because $\sqrt{n^2+1} \approx n$, it follows that $n\sqrt{n^2+1} \approx n^2$. So we will compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2+1}}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} \\
&= 1.
\end{aligned}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ converges.

Problem 23

Problem. Test $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ for convergence or divergence. Identify which test was used.

Solution. Note that $\dfrac{\sqrt[3]{n}}{n} = \dfrac{1}{n^{2/3}}$. Therefore, this is a p-series with $p = \frac{2}{3} < 1$. Therefore, the series diverges.

Problem 25

Problem. Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ for convergence or divergence. Identify which test was used.

Solution. Each term $\dfrac{1}{5^n+1}$ is slightly smaller than $\dfrac{1}{5^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a convergent geometric series. Use the Direct Comparison Test or the Limit Comparison Test.

$$\begin{aligned} \frac{1}{5^n+1} &< \frac{1}{5^n} \\ 5^n &< 5^n + 1 \\ 0 &< 1. \end{aligned}$$

The steps are logically reversible, so we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ converges.

Problem 26

Problem. Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3-8}$ for convergence or divergence. Identify which test was used.

Solution. For large n, $n^3 - 8 \approx n^3$, but the Direct Comparison Test will fail because, although $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, the terms $\displaystyle\frac{1}{n^3-8}$ are larger than the terms $\displaystyle\frac{1}{n^3}$. Instead, use the Limit Comparison Test.

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{n^3-8}}{\frac{1}{n^3}} &= \lim_{n\to\infty} \frac{n^3}{n^3-8} \\ &= \lim_{n\to\infty} \frac{1}{1-\frac{8}{n^3}} \\ &= 1. \end{aligned}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3-8}$ converges.

Problem 27

Problem. Test $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$ for convergence or divergence. Identify which test was used.

Solution. For large n, each term $\displaystyle\frac{2n}{3n-2}$ is approximately $\frac{2}{3}$. Use the Divergence Test.

$$\begin{aligned} \lim_{n\to\infty} \frac{2n}{3n-2} &= \lim_{n\to\infty} \frac{2}{3-\frac{2}{n}} \\ &= \frac{2}{3} \\ &\neq 0. \end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2n}{3n-2}$ diverges.