Monday, November 30, 2015

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Problem 5

Problem. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$.

Solution. The series is alternating with $a_n = \frac{1}{n+1}$. It is clear that $a_{n+1} < a_n$ and that $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Therefore, by the Alternating Series Test, the series converges.

Problem 6

Problem. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3n+2}$.

Solution. Use the Divergence Test. $\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} \neq 0$. Therefore, the series diverges.

Problem 7

Problem. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$.

Solution. This is a geometric series with $r = -\frac{1}{3}$. Because $|r| < 1$, the series converges. In fact, it converges to $-\frac{1}{4}$.

Problem 10

Problem. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+5}$.

Solution. This is an alternating series with $a_n = \frac{n}{n^2+5}$. Check that $a_{n+1} < a_n$.

$$\begin{aligned} a_{n+1} &< a_n \\ \frac{n+1}{(n+1)^2+5} &< \frac{n}{n^2+5} \\ (n+1)(n^2+5) &< n((n+1)^2+5) \\ n^3+n^2+5n+5 &< n^3+2n^2+6n \\ 5 &< n^2+n, \end{aligned}$$

which is true for all $n \geq 2$ (which is sufficient). Now check that $\lim_{n\to\infty} a_n = 0$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{n^2+5} \\ &= \lim_{n\to\infty} \frac{1}{n+\frac{5}{n}} \\ &= 0.\end{aligned}$$

Therefore, the series converges.

Problem 13

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Solution. This is an alternating series with $a_n = \dfrac{1}{\sqrt{n}}$. It is clear that $a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges.

Problem 16

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$.

Solution. This is an alternating series with $a_n = \dfrac{\ln(n+1)}{n+1}$. Check that $a_{n+1} < a_n$.

$$\begin{aligned}a_{n+1} &< a_n \\ \frac{\ln(n+2)}{n+2} &< \frac{\ln(n+1)}{n+1} \\ (n+1)\ln(n+2) &< (n+2)\ln(n+1).\end{aligned}$$

Hmmm...finishing that argument could be tricky, like *real* tricky. Let's try a different approach. Let $f(x) = \dfrac{\ln x}{x}$ and show that $f(x)$ is decreasing for all x greater than some real number.

$$\begin{aligned}f'(x) &= \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} \\ &= \frac{1-\ln x}{x^2}.\end{aligned}$$

Clearly, $f'(x) < 0$ whenever $1 - \ln x < 0$, which is whenever $x > e$. Therefore, the terms of the series are decreasing whenever $n+1 > e$, i.e., whenever $n > e - 1 \approx 1.7$.

Now check that $\lim_{n\to\infty} a_n = 0$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\ln(n+1)}{n+1} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1} \text{ (L'Hôpital's Rule)} \\ &= 0.\end{aligned}$$

Therefore, the series converges.

Problem 23

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

Solution. This is an alternating series with $a_n = \dfrac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$. Rewrite a_n as $\left(\frac{1}{1}\right)\left(\frac{2}{3}\right)\left(\frac{3}{5}\right)\cdots\left(\frac{n}{2n-1}\right)$. This makes it clear that $a_{n+1} = \left(\frac{n+1}{2n+1}\right) a_n$. Because $\dfrac{n+1}{2n+1} < 1$, it follows that $a_{n+1} < a_n$.

Let's postpone this problem to the section on the Ratio Test, where it will be *much* easier to solve.

Problem 24

Problem. Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$.

Solution. Let's postpone this problem to the section on the Ratio Test, where it will be *much* easier to solve.

Problem 27

Problem. Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 5}{n!}$ by using the first six terms.

Solution. Calculate

$$\begin{aligned}\sum_{n=1}^{6} \frac{(-1)^n 5}{n!} &= \frac{5}{1!} - \frac{5}{2!} + \frac{5}{3!} - \frac{5}{4!} + \frac{5}{5!} - \frac{5}{6!} \\ &= \frac{5}{1} - \frac{5}{2} + \frac{5}{6} - \frac{5}{24} + \frac{5}{120} - \frac{5}{720} \\ &= 8.59027\ldots\end{aligned}$$

Problem 29

Problem. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n^3}$ by using the first six terms.

Solution. Calculate

$$\begin{aligned}\sum_{n=1}^{6} \frac{(-1)^{n+1}2}{n^3} &= \frac{2}{1^3} - \frac{2}{2^3} + \frac{2}{3^3} - \frac{2}{4^3} + \frac{2}{5^3} - \frac{2}{6^3} \\ &= \frac{2}{1} - \frac{2}{8} + \frac{2}{27} - \frac{2}{64} + \frac{2}{125} - \frac{2}{216} \\ &= 1.79956\ldots\end{aligned}$$

Problem 31

Problem. Use Theorem 9.15 to determine the number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ with an error of less than 0.001.

Solution. We need the remainder R (i.e., the maximum error) to be less than 0.001. We know from Theorem 9.15 that $|R_N| < a_{N+1}$. Thus, we need to find the first term whose value is less than 0.001. We can find such N by solving the inequality

$$\frac{1}{(N+1)^3} < 0.001.$$

$$\begin{aligned}\frac{1}{(N+1)^3} &< 0.001, \\ (N+1)^3 &> 1000, \\ N+1 &> 10, \\ N &> 9.\end{aligned}$$

Therefore, 10 terms suffice. We find that $S_{10} = 0.90174\ldots$.

Problem 35

Problem. Use Theorem 9.15 to determine the number of terms required to approximate the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$ with an error of less than 0.001.

Solution. We need the remainder R (i.e., the maximum error) to be less than 0.001. We know from Theorem 9.15 that $|R_N| < a_{N+1}$. Thus, we need to find the first term whose value is less than 0.001. We can do that by trial and error.

n	a_n
0	$a_0 = \frac{1}{0!} = 1$
1	$a_1 = \frac{1}{1!} = 1$
2	$a_2 = \frac{1}{2!} = 0.5$
3	$a_3 = \frac{1}{3!} = 0.16666\ldots$
4	$a_4 = \frac{1}{4!} = 0.04166\ldots$
5	$a_5 = \frac{1}{5!} = 0.00833\ldots$
6	$a_6 = \frac{1}{6!} = 0.00138\ldots$
7	$a_7 = \frac{1}{7!} = 0.00019\ldots$

Therefore, 7 terms suffice. We find that $S_7 = 0.36785\ldots$.