Monday, December 7, 2015

p. 644: 26, 27, 28, 30, 41, 42, 44

Problem 26

Problem. Find the 4th Taylor polynomial centered at $c = 2$ of the function $f(x) = \dfrac{1}{x^2}$.

Solution. The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$\frac{f^{(n)}(2)}{n!}$
0	x^{-2}	$\frac{1}{2^2}$	$\frac{1}{2^2}$
1	$-2!x^{-3}$	$-\frac{2!}{2^3}$	$-\frac{2}{2^3}$
2	$3!x^{-4}$	$\frac{3!}{2^4}$	$\frac{3}{2^4}$
3	$-4!x^{-5}$	$-\frac{4!}{2^5}$	$-\frac{4}{2^5}$
4	$5!x^{-6}$	$\frac{5!}{2^6}$	$\frac{5}{2^6}$

The polynomial is

$$\begin{aligned} P_4(x) &= \frac{1}{2^2} - \frac{2(x-2)}{2^3} + \frac{3(x-2)^2}{2^3} - \frac{4(x-2)^3}{2^4} + \frac{5(x-2)^4}{2^6} \\ &= \tfrac{1}{4} - \tfrac{1}{4}(x-2) + \tfrac{3}{8}(x-2)^2 - \tfrac{1}{4}(x-2)^3 + \tfrac{5}{32}(x-2)^4. \end{aligned}$$

Problem 27

Problem. Find the 3rd Taylor polynomial centered at $c = 4$ of the function $f(x) = \sqrt{x}$.

Solution. The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(4)$	$\frac{f^{(n)}(4)}{n!}$
0	$x^{1/2}$	2	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$
2	$\frac{1}{2^2}!x^{-3/2}$	$\frac{1}{2^5}$	$\frac{1}{2^5 \cdot 2!}$
3	$\frac{1\cdot 3}{2^3}x^{-5/2}$	$\frac{1\cdot 3}{2^8}$	$\frac{1\cdot 3}{2^8 \cdot 3!}$

The polynomial is

$$\begin{aligned} P_4(x) &= 2 + \frac{x-4}{2^2} + \frac{(x-4)^2}{2^5 \cdot 2!} + \frac{(1\cdot 3)(x-4)^3}{2^8 \cdot 3!} \\ &= 2 + \tfrac{1}{4}(x-4) + \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3. \end{aligned}$$

Problem 28

Problem. Find the 3rd Taylor polynomial centered at $c = 8$ of the function $f(x) = \sqrt[3]{x}$.

Solution. The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(8)$	$\frac{f^{(n)}(8)}{n!}$
0	$x^{1/3}$	2	2
1	$\frac{1}{3}x^{-2/3}$	$\frac{1}{3\cdot 2^2}$	$\frac{1}{3\cdot 2^2}$
2	$-\frac{2}{3^2}!x^{-5/3}$	$\frac{2}{3^2\cdot 2^5}$	$\frac{2}{3^2\cdot 2^5\cdot 2!}$
3	$\frac{2\cdot 5}{3^3}x^{-8/3}$	$\frac{2\cdot 5}{3^3\cdot 2^8}$	$\frac{2\cdot 5}{3^3\cdot 2^8\cdot 3!}$

The polynomial is

$$\begin{aligned} P_4(x) &= 2 + \frac{x-8}{2^2} - \frac{2(x-8)^2}{3^2\cdot 2^5\cdot 2!} + \frac{(2\cdot 5)(x-8)^3}{3^3\cdot 2^8\cdot 3!} \\ &= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 + \tfrac{5}{10368}(x-8)^3. \end{aligned}$$

Problem 30

Problem. Find the 2nd Taylor polynomial centered at $c = \pi$ of the function $f(x) = x^2 \cos x$.

Solution. First, we need the first two derivatives of $x^2 \cos x$.

$$\begin{aligned} f(x) &= x^2 \cos x, \\ f'(x) &= (2x)(\cos x) + (x^2)(-\sin x) \\ &= 2x\cos x - x^2 \sin x, \\ f''(x) &= ((2)(\cos(x)) + (2x)(-\sin x)) - \big((2x)(\sin x) + (x^2)(\cos x)\big) \\ &= (2 - x^2)\cos x - 4x \sin x. \end{aligned}$$

The table of coefficients (using the facts that $\cos\pi = -1$ and $\sin\pi = 0$):

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$	$\frac{f^{(n)}(\pi)}{n!}$
0	$x^2\cos x$	$-\pi^2$	$-\pi^2$
1	$2x\cos x - x^2\sin x$	-2π	-2π
2	$(2-x^2)\cos x - 4x\sin x$	$\pi^2 - 2$	$\frac{\pi^2-2}{2!}$

The polynomial is

$$\begin{aligned} P_4(x) &= -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2-2)(x-\pi)^2}{2!} \\ &= -\pi^2 - 2\pi(x-\pi) + \tfrac{(\pi^2-2)}{2}(x-\pi)^2. \end{aligned}$$

Problem 41

Problem. Approximate the function $f(x) = e^{4x}$ at $x = \frac{1}{4}$ using the polynomial found in Exercise 13.

Solution. The polynomial found in Exercise 13 is

$$P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4.$$

Therefore,

$$\begin{aligned} P_4(\tfrac{1}{4}) &= 1 + 4\left(\frac{1}{4}\right) + 8\left(\frac{1}{4}\right)^2 + \frac{32}{3}\left(\frac{1}{4}\right)^3 + \frac{32}{3}\left(\frac{1}{4}\right)^4 \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \\ &= 2.708333\cdots \end{aligned}$$

Problem 42

Problem. Approximate the function $f(x) = x^2e^{-x}$ at $x = \frac{1}{5}$ using the polynomial found in Exercise 20.

Solution. The polynomial found in Exercise 20 is

$$P_4(x) = x^2 - x^3 + \frac{1}{2}x^4.$$

Therefore,

$$\begin{aligned} P_4(\tfrac{1}{5}) &= \left(\frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^3 + \frac{1}{2}\left(\frac{1}{5}\right)^4 \\ &= \frac{1}{25} - \frac{1}{125} + \frac{1}{1250} \\ &= 0.0328. \end{aligned}$$

Problem 44

Problem. Approximate the function $f(x) = x^2\cos x$ at $x = \frac{7\pi}{8}$ using the polynomial found in Exercise 30.

Solution. The polynomial found in Exercise 30 is

$$P_2(x) = -\pi^2 - 2\pi(x - \pi) + \frac{(\pi^2 - 2)}{2}(x - \pi)^2.$$

Therefore,

$$\begin{aligned}
P_2(\tfrac{7\pi}{8}) &= -\pi^2 - 2\pi\left(\frac{7\pi}{8} - \pi\right) + \frac{(\pi^2-2)}{2}\left(\frac{7\pi}{8} - \pi\right)^2 \\
&= -\pi^2 - 2\pi\left(-\frac{\pi}{8}\right) + \frac{(\pi^2-2)}{2}\left(-\frac{\pi}{8}\right)^2 \\
&= -\pi^2 + \frac{\pi^2}{4} + \frac{(\pi^2-2)\pi^2}{128} \\
&= -\frac{49\pi^2}{64} + \frac{\pi^4}{128} \\
&= -6.7954\ldots
\end{aligned}$$