Tuesday, September 8, 2015

Page 352

Problem 71

Problem. Find the extrema and the points of inflection of the function $f(x) = \dfrac{e^x + e^{-x}}{2}$.

Solution. First, find $f'(x)$.

$$f'(x) = \frac{e^x - e^{-x}}{2}.$$

Now solve $f'(x) = 0$.

$$\begin{aligned} \frac{e^x - e^{-x}}{2} &= 0 \\ e^x - e^{-x} &= 0 \\ e^{2x} - 1 &= 0 \\ e^{2x} &= 1 \\ 2x &= \ln 1 = 0 \\ x &= 0. \end{aligned}$$

Use test points $x = -1$ and $x = 1$. We find that $f'(-1) = \dfrac{e^{-1} - e^1}{2} < 0$ and $f'(1) = \dfrac{e^1 - e^{-1}}{2} > 0$. Therefore, a minimum of $f(0) = 1$ occurs when $x = 0$. Next, find $f''(x)$.

$$f''(x) = \frac{e^x + e^{-x}}{2}.$$

Solve $f''(x) = 0$. We get

$$\begin{aligned} \frac{e^x + e^{-x}}{2} &= 0 \\ e^x + e^{-x} &= 0 \\ e^{2x} + 1 &= 0 \\ e^{2x} &= -1. \end{aligned}$$

That last equation has no solution because $e^{2x} > 0$ for all x. Therefore, the function has no inflection point.

Problem 72

Problem. Find the extrema and the points of inflection of the function $f(x) = \dfrac{e^x - e^{-x}}{2}$.

Solution. First, find $f'(x)$.

$$f'(x) = \frac{e^x + e^{-x}}{2}.$$

Now solve $f'(x) = 0$.

$$\frac{e^x + e^{-x}}{2} = 0.$$

We saw in the previous problem that this equation has no solution. Therefore, the function has neither a maximum nor a minimum. Next, find $f''(x)$.

$$f''(x) = \frac{e^x - e^{-x}}{2}.$$

Solve $f''(x) = 0$. We get

$$\begin{aligned} \frac{e^x - e^{-x}}{2} &= 0 \\ x &= 0. \end{aligned}$$

(as we saw in the previous problem.) Use test points $x = -1$ and $x = 1$. We find that $f''(-1) = \dfrac{e^{-1} - e^1}{2} < 0$ and $f''(1) = \dfrac{e^1 - e^{-1}}{2} > 0$. Therefore, an inflection point occurs when $x = 0$.

Problem 93

Problem. Find the indefinite integral $\displaystyle\int e^{2x-1}\,dx$.

Solution. Let $u = 2x - 1$ and $du = 2\,dx$. Then

$$\begin{aligned} \int e^{2x-1}\,dx &= \frac{1}{2}\int 2e^{2x-1}\,dx \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{2x-1} + C. \end{aligned}$$

Problem 95

Problem. Find the indefinite integral $\int x^2 e^{x^3}\, dx$.

Solution. Let $u = x^3$ and $du = 3x^2\, dx$. Then

$$\begin{aligned}\int x^2 e^{x^3}\, dx &= \frac{1}{3}\int 3x^2 e^{x^3}\, dx \\ &= \frac{1}{3}\int e^u\, du \\ &= \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{x^3} + C.\end{aligned}$$

Problem 97

Problem. Find the indefinite integral $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

Solution. Let $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}}\, dx$. Then

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx &= 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\, dx \\ &= 2\int e^u\, du \\ &= 2e^u + C \\ &= 2e^{\sqrt{x}} + C.\end{aligned}$$

Problem 99

Problem. Find the indefinite integral $\int \frac{e^{-x}}{1+e^{-x}}\, dx$.

Solution. Let $u = 1 + e^{-x}$ and $du = -e^{-x}\, dx$. Then

$$\begin{aligned}\int \frac{e^{-x}}{1+e^{-x}}\, dx &= -\int \frac{-e^{-x}}{1+e^{-x}}\, dx \\ &= -\ln|1+e^{-x}| + C.\end{aligned}$$

(I did not even bother to make the substitution once I saw that the numerator was the derivative of the denominator.)

Problem 103

Problem. Find the indefinite integral $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$.

Solution. Let $u = e^x - e^{-x}$ and $du = (e^x + e^{-x})\, dx$. Then

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \ln |e^x - e^{-x}| + C.$$

(Again, I did not even bother to make the substitution once I saw that the numerator was the derivative of the denominator.) Let $u = 1 + e^{-x}$ and $du = -e^{-x}\, dx$.

Problem 105

Problem. Find the indefinite integral $\int \frac{5 - e^x}{e^{2x}}\, dx$.

Solution. The denominator is a monomial, so let's try dividing by it to break the function up into two fractions.

$$\begin{aligned}
\int \frac{5 - e^x}{e^{2x}}\, dx &= \int \left(\frac{5}{e^{2x}} - \frac{e^x}{e^{2x}}\right)\, dx \\
&= \int \left(5e^{-2x} - e^{-x}\right)\, dx \\
&= 5\left(-\frac{1}{2}\right) e^{-2x} + e^{-x} + C \\
&= -\frac{5}{2}e^{-2x} + e^{-x} + C.
\end{aligned}$$

Problem 109

Problem. Evaluate the definite integral $\int_0^1 e^{-2x}\, dx$.

Solution.

$$\begin{aligned}
\int_0^1 e^{-2x}\, dx &= -\frac{1}{2}\left[e^{-2x}\right]_0^1 \\
&= -\frac{1}{2}\left(e^{-2} - e^0\right) \\
&= -\frac{1}{2}\left(e^{-2} - 1\right).
\end{aligned}$$

Problem 113

Problem. Evaluate the definite integral $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$.

Solution. Let $u = \frac{3}{x}$ and $du = -\frac{3}{x^2}\,dx$. Also, $u(1) = 3$ and $u(3) = 1$.

$$\begin{aligned}\int_1^3 \frac{e^{3/x}}{x^2}\,dx &= -\frac{1}{3}\int_1^3 \frac{(-3)e^{3/x}}{x^2}\,dx \\ &= -\frac{1}{3}\int_3^1 e^u\,du \\ &= -\frac{1}{3}\left[e^u\right]_3^1 \\ &= -\frac{1}{3}\left(e^1 - e^3\right) \\ &= \frac{e^3 - e}{3}.\end{aligned}$$

Problem 116

Problem. Evaluate the definite integral $\int_0^1 \frac{e^x}{5 - e^x}\,dx$.

Solution. Let $u = 5 - e^x$ and $du = -e^x$. We see that the numerator is nearly the derivative of the denominator (adjust by -1), so we have

$$\begin{aligned}\int_0^1 \frac{e^x}{5 - e^x}\,dx &= -\int_0^1 \frac{-e^x}{5 - e^x}\,dx \\ &= -\left[\ln|5 - e^x|\right]_0^1 \\ &= -\left(\ln|5 - e| - \ln|5 - 1|\right) \\ &= \ln 4 - \ln(5 - e).\end{aligned}$$

Problem 141

Problem. The position function of a particle moving along the x-axis is $x(t) = Ae^{kt} + Be^{-kt}$, where A, B, and k are positive constants.

(a) During what times t is the particle closest to the origin?

(b) Show that the acceleration of the particle is proportional to the position of the particle. What is the constant of proportionality?

Solution. (a) The function $x(t)$ is always positive, so to find when it is nearest the origin is to find the value(s) of t that minimize $x(t)$.

First, find $x'(t)$ and then solve $x'(t) = 0$.

$$x'(t) = Ake^{kt} - Bke^{-kt}.$$

Now solve $x'(t) = 0$.

$$\begin{aligned}
Ake^{kt} - Bke^{-kt} &= 0 \\
Ae^{kt} - Be^{-kt} &= 0 \\
Ae^{kt} &= Be^{-kt} \\
Ae^{2kt} &= B \\
e^{2kt} &= \frac{B}{A} \\
2kt &= \ln \frac{B}{A} = \ln B - \ln A \\
t &= \frac{\ln B - \ln A}{2k}.
\end{aligned}$$

(b) To find the acceleration, we need $x''(t)$.

$$\begin{aligned}
x''(t) &= Ak^2e^{kt} + Bk^2e^{-kt} \\
&= k^2 \left(Ae^{kt} + Be^{-kt}\right) \\
&= k^2 x(t).
\end{aligned}$$

The constant of proportionality is k^2.

Problem 142

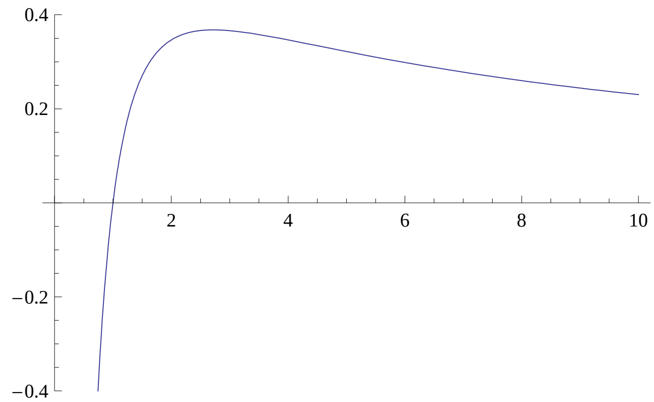
Problem. Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \leq A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

Solution. (a) The graph is



(b) It appears that the graph has a maximum between 2 and 3. To the right of the maximum, the function is decreasing. We will find the maximum.

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} \\ &= \frac{1 - \ln x}{x^2}. \end{aligned}$$

Solve $f'(x) = 0$.

$$\begin{aligned} \frac{1 - \ln x}{x^2} &= 0 \\ 1 - \ln x &= 0 \\ \ln x &= 1 \\ x &= e. \end{aligned}$$

So a maximum of $f(e) = \frac{\ln e}{e} = \frac{1}{e}$ occurs at $x = e$. If $e \leq A < B$, then $f(A) > f(B)$, so

$$\begin{aligned} \frac{\ln A}{A} &> \frac{\ln B}{B} \\ B \ln A &> A \ln B \\ \ln A^B &> \ln B^A \\ A^B &> B^A. \end{aligned}$$

(c) We know that $e < \pi$. It then follows from part (b) that $e^\pi > \pi^e$. Cool!