Wednesday, September 9, 2015

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Problem 1

Problem. Evaluate the expression $\log_2 \frac{1}{8}$ without using a calculator.

Solution. We know that $\frac{1}{8} = 2^{-3}$. Therefore, $\log_2 \frac{1}{8} = -3$.

Problem 6

Problem. Write the exponential equations as logarithmic equations.

(a) $27^{2/3} = 9$.

(b) $16^{3/4} = 8$.

Solution. (a) The base is 27 and the exponent (logarithm) is 2/3. So the logarithmic equation is

$$\log_{27} 9 = \frac{2}{3}.$$

(b) The base is 16 and the exponent (logarithm) is 3/4. So the logarithmic equation is

$$\log_{16} 8 = \frac{3}{4}.$$

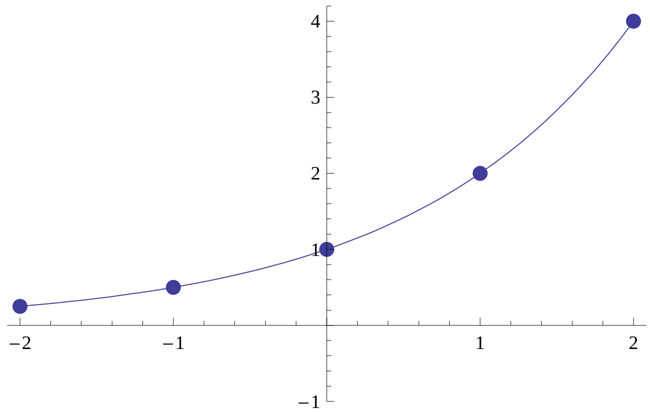
Problem 9

Problem. Sketch the graph of the function $y = 2^x$ by hand.

Solution. Plot a few points.

x	y
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$

Now draw the graph.



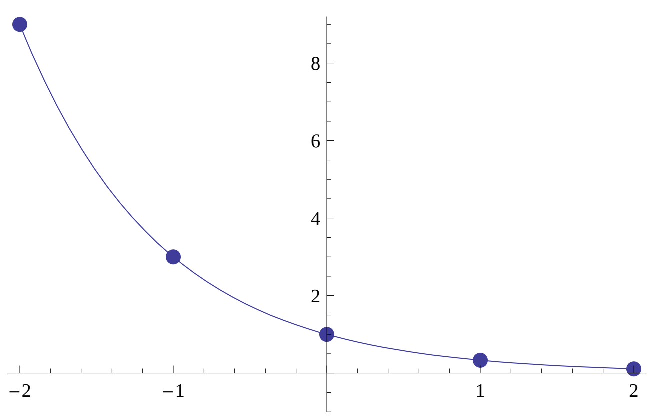
Problem 11

Problem. Sketch the graph of the function $y = \left(\frac{1}{3}\right)^x$ by hand.

Solution. Plot a few points.

x	y
-2	$\left(\frac{1}{3}\right)^{-2} = 9$
-1	$\left(\frac{1}{3}\right)^{-1} = 3$
0	$\left(\frac{1}{3}\right)^{0} = 1$
1	$\left(\frac{1}{3}\right)^{1} = \frac{1}{3}$
2	$\left(\frac{1}{3}\right)^{2} = \frac{1}{9}$

Now draw the graph.



Problem 25

Problem. Solve the equation $3^{2x} = 75$ accurate to three decimal places.

Solution.

$$
\begin{aligned}
3^{2x} &= 75 \\
2x &= \log_3 75 \\
x &= \frac{1}{2}\log_3 75 \\
&= \frac{\ln 75}{2\ln 3} \\
&= 1.965.
\end{aligned}
$$

Problem 41

Problem. Find the derivative of the function $f(x) = 4^x$.

Solution.

$$f'(x) = 4^x \ln 4.$$

Problem 47

Problem. Find the derivative of the function $y = \log_4(5x+1)$.

Solution. Use the rule for logarithmic functions and the Chain Rule.

$$
\begin{aligned}
y' &= \frac{1}{(5x+1)\ln 4} \cdot 5 \\
&= \frac{5}{(5x+1)\ln 4}
\end{aligned}
$$

Problem 53

Problem. Find the derivative of the function $f(x) = \log_2 \dfrac{x^2}{x-1}$.

Solution. First, rewrite the function as $f(x) = 2\log_2 x - \log_2(x-1)$. Then differentiate.

$$f'(x) = 2 \cdot \frac{1}{x \ln 2} - \frac{1}{(x-1)\ln 2}.$$

Problem 60

Problem. Find the equation of the tangent line to the graph of the function $y = 5^{x-2}$ at the point $(2, 1)$.

Solution. The derivative is

$$y' = 5^{x-2} \ln 5.$$

The slope of the tangent line at $x = 2$ is $y'(2) = 5^0 \ln 5 = \ln 5$. Now use the point-slope form to get the equation of the tangent line.

$$\begin{aligned} y - 1 &= (\ln 5)(x - 2) \\ y &= 1 + (x - 2) \ln 5. \end{aligned}$$

Problem 70

Problem. Find the equation of the tangent line to the graph of the function $y = x^{1/x}$ at the point $(1, 1)$.

Solution. Use logarithmic differentiation to find y'.

$$\begin{aligned} \ln y &= \ln x^{1/x} \\ &= \frac{1}{x} \cdot \ln x \\ &= \frac{\ln x}{x} \\ \frac{y'}{y} &= \frac{\frac{1}{x} \cdot x - (\ln x) \cdot 1}{x^2} \\ &= \frac{1 - \ln x}{x^2} \\ y' &= y \cdot \frac{1 - \ln x}{x^2} \\ &= x^{1/x} \cdot \frac{1 - \ln x}{x^2}. \text{ (Wow!)} \end{aligned}$$

Now find the slope:

$$\begin{aligned} y'(1) &= 1^1 \cdot \frac{1 - \ln 1}{1^2} \\ &= 1. \end{aligned}$$

Then the equation of the line is

$$y - 1 = 1 \cdot (x - 1),$$
$$y = x.$$

Problem 71

Problem. Find the indefinite integral $\int 3^x \, dx$.

Solution.

$$\int 3^x \, dx = \frac{3^x}{\ln 3} + C.$$

Problem 73

Problem. Find the indefinite integral $\int (x^2 + 2^{-x}) \, dx$.

Solution.

$$\begin{aligned} \int (x^2 + 2^{-x}) \, dx &= \int x^2 \, dx + \int 2^{-x} \, dx \\ &= \frac{1}{3}x^3 + \int 2^{-x} \, dx. \end{aligned}$$

For the second integral, we may want to use the substitution $u = -x$, $du = -dx$.

$$\begin{aligned} \int 2^{-x} \, dx &= -\int (-2^{-x}) \, dx \\ &= -\int 2^u \, du \\ &= -\frac{2^u}{\ln 2} \\ &= -\frac{2^{-x}}{\ln 2}. \end{aligned}$$

So the answer is

$$\int (x^2 + 2^{-x}) \, dx = \frac{1}{3}x^3 - \frac{2^{-x}}{\ln 2} + C.$$

Problem 75

Problem. Find the indefinite integral $\int x(5^{-x^2})\,dx$.

Solution. Let $u = -x^2$ and $du = -2x\,dx$. Then

$$\begin{aligned}
\int x(5^{-x^2})\,dx &= -\frac{1}{2}\int (-2x)(5^{-x^2})\,dx \\
&= -\frac{1}{2}\int 5^u\,du \\
&= -\frac{1}{2}\cdot\frac{5^u}{\ln 5} + C \\
&= -\frac{5^{-x^2}}{2\ln 5} + C.
\end{aligned}$$