Math 142 **Test #2 Answers** **Fall 2011**

1. (8 pts)

   (a) (2 pts) $\sinh 1 = \frac{1}{2}\left(e - e^{-1}\right)$

   (b) (3 pts) Because $\tanh x = \dfrac{\sinh x}{\cosh x}$, you might begin by calculating $\sinh \ln 2$ and $\cosh \ln 2$.

$$\begin{aligned}
\sinh \ln 2 &= \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) \\
&= \frac{1}{2}\left(2 - \frac{1}{2}\right) \\
&= \frac{3}{4}. \\
\cosh \ln 2 &= \frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) \\
&= \frac{1}{2}\left(2 + \frac{1}{2}\right) \\
&= \frac{5}{4}.
\end{aligned}$$

   Therefore,

$$\begin{aligned}
\tanh(\ln 2) &= \frac{3/4}{5/4} \\
&= \frac{3}{5}.
\end{aligned}$$

   (c) (3 pts) $\dfrac{d}{dx}(\sinh x) = \cosh x$

2. (10 pts) Find the following indefinite integrals.

   (a) (5 pts) Use the formula sheet.

$$\int \frac{1}{25 - x^2}\,dx = \frac{1}{10}\ln\left|\frac{5+x}{5-x}\right| + C.$$

   (b) (5 pts) Use the formula sheet.

$$\begin{aligned}
\int \frac{2}{3x\sqrt{9+x^2}}\,dx &= -\frac{2}{3}\cdot\frac{1}{3}\ln\left(\frac{3+\sqrt{9+x^2}}{|x|}\right) + C \\
&= -\frac{2}{9}\ln\left(\frac{3+\sqrt{9+x^2}}{|x|}\right) + C.
\end{aligned}$$

3. (14 pts) Given the differential equation

$$\frac{dy}{dx} = 2x(y+2),$$

(a) (10 pts) Find the general solution.

Separate the variables and integrate both sides.

$$\begin{aligned}
\frac{dy}{y+2} &= 2x\ dx \\
\int \frac{dy}{y+2} &= \int 2x\ dx \\
\ln|y+2| &= x^2 + C \\
|y+2| &= e^{x^2+C} \\
&= e^C e^{x^2} \\
&= C_1 e^{x^2} \quad \text{where } C_1 = e^C > 0 \\
y+2 &= \pm C_1 e^{x^2} \\
&= C_2 e^{x^2} \quad \text{where } C_2 = \pm C_1 \neq 0 \\
y &= -2 + C_2 e^{x^2}.
\end{aligned}$$

Then you can check that if $C_2 = 0$, we get $y = -2$ which is a solution. Therefore, the general solution is

$$y = -2 + C_2 e^{x^2},$$

where C_2 is an arbitrary constant.

(b) (4 pts) Find the particular solution that satisfies the initial condition $y(0) = 4$.

Let $x = 0$ and $y = 4$ and solve for C_2.

$$\begin{aligned}
4 &= -2 + C_2 e^0 \\
&= -2 + C_2 \\
C_2 &= 6.
\end{aligned}$$

The particular solution is $y = -2 + 6e^{x^2}$.

4. (15 pts)

The height of the graph is x, so the area of a cross section is πx^2. Then the volume is

$$
\begin{aligned}
V &= \int_2^4 \pi x^2 \, dx \\
&= \pi \left[\frac{1}{3} x^3 \right]_2^4 \\
&= \pi \left(\frac{64}{3} - \frac{8}{3} \right) \\
&= \frac{56\pi}{3}.
\end{aligned}
$$

5. (12 pts)

   The radius of each shell is $r = x$ and the height is $h = 9 - x^2$, so the area of the shell is $A = 2\pi rh = 2\pi x(9 - x^2) = 2\pi(9x - x^3)$. Also, the function meets the x-axis at $x = 3$. So the volume of the solid is

$$
\begin{aligned}
V &= \int_0^3 2\pi(9x - x^3) \, dx \\
&= 2\pi \left[\frac{9}{2} x^2 - \frac{1}{4} x^4 \right]_0^3 \\
&= 2\pi \left(\frac{81}{2} - \frac{81}{4} \right) \\
&= 2\pi \left(\frac{81}{4} \right) \\
&= \frac{81\pi}{2}.
\end{aligned}
$$

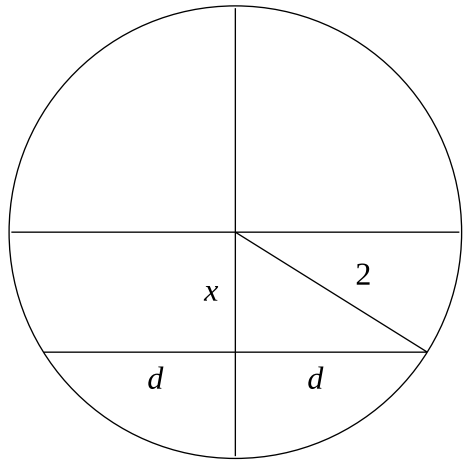
6. (8 pts)

   The derivative is $y' = 3x^2$, so the integral is

$$
\int_0^1 \sqrt{1 + (3x^2)^2} \, dx = \int_0^1 \sqrt{1 + 9x^4} \, dx.
$$

7. (15 pts)

   A horizontal cross section of the tank is rectangular with one side 10 and the other side a chord or the circular end. We need to find the length of the chord at height x. It will be best to let the center of the circle be at the origin. Then the ground level is at $x = -2$. (Or $y = -2$ if you like.) So if the water is lifted to a height x, then the distance raised is $x - (-2) = x + 2$. (This has a positive value because $-2 \le x \le 0$.) At the level x, the distance across the circle is $2d = 2\sqrt{4 - x^2}$. (See the diagram.) So the area of the rectangle is $A = 20\sqrt{4 - x^2}$. Now we can set up the integral and find the work done.



$$
\begin{aligned}
W &= \int_{-2}^{0} \rho(x+2) \cdot 20\sqrt{4-x^2}\, dx \\
&= 20\rho \int_{-2}^{0} (x+2)\sqrt{4-x^2}\, dx \\
&= 20\rho \int_{-2}^{0} x\sqrt{4-x^2}\, dx + 40\rho \int_{-2}^{0} \sqrt{4-x^2}\, dx.
\end{aligned}
$$

We can evaluate the first integral by using the substitution $u = 4 - x^2$ and $du = -2x\, dx$. The second integral we realize represents the area of a quarter of a circle of radius 2. Therefore, its value is $\frac{1}{4}\pi \cdot 2^2 = \pi$. We continue with the integration:

$$
\begin{aligned}
W &= 20\rho \int_{-2}^{0} x\sqrt{4-x^2}\, dx + 40\rho \int_{-2}^{0} \sqrt{4-x^2}\, dx \\
&= -10\rho \int_{0}^{4} \sqrt{u}\, du + 40\rho\pi \\
&= -10\rho \left[\frac{2}{3}u^{3/2}\right]_0^4 + 40\rho\pi \\
&= -10\rho \left(\frac{16}{3}\right) + 40\rho\pi \\
&= -\frac{160\rho}{3} + 40\rho\pi \\
&= 40\rho\left(\pi - \frac{4}{3}\right).
\end{aligned}
$$

8. (18 pts)

(a) (10 pts)

You can set up the integral with respect to x and integrate from $x = -3$ to $x = 3$ or with respect to y and integrate from $y = 0$ to $y = 9$. I think it is easier to do it with respect to y. With respect to y, rewrite the function as $x = \sqrt{y}$. (Actually, it is $x = \sqrt{y}$ to the right and $x = -\sqrt{y}$ to the left, but that is no complication.) Then the distance across the region is $2\sqrt{y}$, so the integral for M_x is

$$\begin{aligned} M_x &= \int_0^9 \rho y \cdot 2\sqrt{y}\, dy \\ &= 2\rho \int_0^9 y^{3/2}\, dy \\ &= 2\rho \left[\frac{2}{5} y^{5/2}\right]_0^9 \\ &= 2\rho \left(\frac{2}{5} \cdot 243\right) \\ &= \frac{972\rho}{5}. \end{aligned}$$

If you integrate with respect to x, then you have to consider the difference between $y = 9$ and $y = x^2$ and integrate half the difference between their squares:

$$\begin{aligned} M_x &= \int_{-3}^{3} \rho \frac{1}{2} \left(9^2 - (x^2)^2\right)\, dx \\ &= \frac{\rho}{2} \int_{-3}^{3} \left(81 - x^4\right)\, dx \\ &= \frac{\rho}{2} \left[81x - \frac{1}{5} x^5\right]_{-3}^{3} \\ &= \frac{\rho}{2} \left(\left(243 - \frac{243}{5}\right) - \left(-243 + \frac{243}{5}\right)\right) \\ &= \frac{972\rho}{5}. \end{aligned}$$

(b) (5 pts)

The mass is simply the area times ρ.

$$\begin{aligned} m &= \rho \int_{-3}^{3} (9 - x^2)\, dx \\ &= \rho \left[9x - \frac{1}{3} x^3\right]_{-3}^{3} \\ &= \rho\left((27 - 9) - (-27 + 9)\right) \\ &= 36\rho. \end{aligned}$$

(c) (3 pts) The center of mass is

$$\begin{aligned}(\overline{x}, \overline{y}) &= \left(\frac{M_x}{m}, 0\right) \\ &= \left(\frac{\frac{972\rho}{5}}{36\rho}, 0\right) \\ &= \left(\frac{972\rho}{180\rho}, 0\right) \\ &= \left(\frac{27}{5}, 0\right)\end{aligned}$$