Math 142 **Test #3 Answers** **Fall 2011**

1. (22 pts)

   (a) (10 pts)
   Use L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0}\frac{\arctan x}{x^2} &= \lim_{x\to 0}\frac{\left(\frac{1}{x^2+1}\right)}{2x}\\ &= \lim_{x\to 0}\frac{1}{2x(x^2+1)},\end{aligned}$$

   which is undefined.

   (b) (12 pts)
   Use L'Hôpital's Rule. Take the logarithm of the limit first.

$$\begin{aligned}\ln\left(\lim_{x\to 0^-}(1+x)^{\csc x}\right) &= \lim_{x\to 0^-}\left(\ln(1+x)^{\csc x}\right)\\ &= \lim_{x\to 0^-}\left(\csc x\ln(1+x)\right)\\ &= \lim_{x\to 0^-}\left(\frac{\ln(1+x)}{\sin x}\right)\\ &= \lim_{x\to 0^-}\left(\frac{\left(\frac{1}{1+x}\right)}{\cos x}\right)\\ &= \frac{\left(\frac{1}{1+0}\right)}{\cos 0}\\ &= 1.\end{aligned}$$

   Therefore, $\lim_{x\to 0^-}(1+x)^{\csc x} = e^1 = e$.

2. (10 pts)
   Replace $\cos^2 x$ with $1-\sin^2 x$ and then make the substitution $u = \sin x$, $du = \cos x\, dx$.

$$\begin{aligned}\int \cos^3 x\, dx &= \int (1-\sin^2 x)\cos x\, dx\\ &= \int (1-u^2)\, du\\ &= u - \frac{1}{3}u^3 + C\\ &= \sin x - \frac{1}{3}\sin^3 x + C.\end{aligned}$$

3. (15 pts)

The expression $\sqrt{x^2+16}$ indicates the substitution

$$\begin{aligned} x &= 4\tan\theta \\ dx &= 4\sec^2\theta\ d\theta \\ \sqrt{x^2+16} &= 4\sec\theta. \end{aligned}$$

Next, use the identity $\tan^2\theta = \sec^2\theta - 1$.

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2+16}}\ dx &= \int \frac{(4\tan\theta)^3}{4\sec\theta} \cdot 4\sec^2\theta\ d\theta \\ &= 64\int \tan^3\theta\sec\theta\ d\theta \\ &= 64\int (\sec^2\theta - 1)\cdot\tan\theta\sec\theta\ d\theta. \end{aligned}$$

Finally, let $u = \sec\theta$, $du = \tan\theta\sec\theta\ d\theta$ and integrate.

$$\begin{aligned} 64\int (\sec^2\theta - 1)\cdot\tan\theta\sec\theta\ d\theta &= 64\int (u^2-1)\ du \\ &= 64\left(\frac{1}{3}u^3 - u\right) \\ &= 64\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) \\ &= \frac{64}{3}(x^2+16)^{3/2} - 64(x^2+16)^{1/2}. \end{aligned}$$

4. (18 pts)

In might be simpler to find the indefinite integral first, and then incorporate the limits of integration. The integration will require integration by parts twice. First time, let $u = e^x$ and $dv = \sin x\ dx$. Then $du = e^x\ dx$ and $v = -\cos x$.

$$\int e^x \sin x\ dx = -e^x\cos x + \int e^x\cos x\ dx.$$

Now do it again, this time letting $u = e^x$ and $dv = \cos x\ dx$ and $du = e^x\ dx$ and $v = \sin x$.

$$\begin{aligned} \int e^x\sin x\ dx &= -e^x\cos x + e^x\sin x - \int e^x\sin x\ dx. \\ 2\int e^x\sin x\ dx &= -e^x\cos x + e^x\sin x \\ \int e^x\sin x\ dx &= \frac{e^x}{2}\left(\sin x - \cos x\right). \end{aligned}$$

Then

$$\begin{aligned}
\int_0^{\pi/4} e^x \sin x \, dx &= \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^{\pi/4} \\
&= \left(\frac{e^{\pi/4}}{2}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - \frac{e^0}{2}(\sin 0 - \cos 0)\right) \\
&= \left(\frac{e^{\pi/4}}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}(0-1)\right) \\
&= \frac{1}{2}.
\end{aligned}$$

5. (24 pts)

(a) (12 pts) $\displaystyle\int \frac{x}{(x-2)^3}\,dx$

Let

$$\begin{aligned}
\frac{x}{(x-2)^3} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}. \\
x &= A(x-2)^2 + B(x-2) + C.
\end{aligned}$$

Let $x = 2$ and get $2 = C$. Then you could differentiate twice and substitute $x = 2$ each time (easy) or you could choose two different values of x. Let $x = 1$ and get $1 = A - B + C$, so $A - B = -1$. Then let $x = 3$ and get $3 = A + B + C$, so $A + B = 1$. It follows that $A = 0$ and $B = 1$. Now integrate.

$$\begin{aligned}
\int \frac{x}{(x-2)^3}\,dx &= \int \frac{1}{(x-2)^2}\,dx + \int \frac{2}{(x-2)^3}\,dx \\
&= -\frac{1}{x-2} - \frac{1}{(x-2)^2} + C.
\end{aligned}$$

[Had we differentiated, we would have gotten $1 = 2A(x-2)+B$. Let $x = 2$ and get $B = 1$. Then differentiate again: $0 = 2A$ and get $A = 0$. Bingo!]

(b) (12 pts) $\displaystyle\int \frac{x^2-2x-1}{(x+1)(x^2+1)}\,dx$

Let

$$\begin{aligned}
\frac{x^2-2x-1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1}. \\
x^2 - 2x - 1 &= A(x^2+1) + (Bx+C)(x+1).
\end{aligned}$$

Let $x = -1$ and get $2 = 2A$, so $A = 1$. Then let $x = 0$ and get $-1 = A+C$, so $C = -2$. And let $x = 1$ and get $-2 = 2A + 2B + 2C$, so $B = 0$. Now integrate.

$$\begin{aligned}
\int \frac{x^2-2x-1}{(x+1)(x^2+1)}\,dx &= \int \frac{1}{x+1}\,dx - \int \frac{2}{x^2+1}\,dx \\
&= \ln|x+1| - 2\arctan x + C.
\end{aligned}$$

6. (12 pts)

We must divide this integral into two integrals at some value of x. We might as well divide it at $x = 0$. Then treat each part as an improper integral.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to-\infty} [\arctan x]_t^0 + \lim_{t\to\infty} [\arctan x]_0^t \\
&= \lim_{t\to-\infty} (\arctan 0 - \arctan t) + \lim_{t\to\infty} (\arctan t - \arctan 0) \\
&= \lim_{t\to-\infty} (-\arctan t) + \lim_{t\to\infty} (\arctan t) \\
&= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} \\
&= \pi.
\end{aligned}$$