Math 142 Test #1 Answers Spring 2008

$(41, 58, 73, 89.3, 97)$, $\overline{x} = 72.5$.

1. (35 pts) $(19, 27, 29, 32, 35)$, $\overline{x} = 29.0$.

   (a) (7 pts) $(6, 7, 7, 7, 7)$, $\overline{x} = 7.0$.
   $f'(x) = e^x$.

   (b) (7 pts) $(0, 1, 7, 7, 7)$, $\overline{x} = 4.6$.
   Use the quotient rule:

$$f'(x) = \frac{(1+e^x)\cdot \frac{d}{dx}(e^x) - \frac{d}{dx}(1+e^x)\cdot(e^x)}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.$$

   (c) (7 pts) $(4, 7, 7, 7, 7)$, $\overline{x} = 6.6$.
   Use the product rule:

$$f'(x) = \frac{d}{dx}(x^2)\cdot \ln(x^2+1) + x^2 \cdot \frac{d}{dx}\left(\ln(x^2+1)\right) = (2x)\ln(x^2+1) + (x^2)\left(\frac{2x}{x^2+1}\right).$$

   (d) (7 pts) $(2, 3.8, 7, 7, 7)$, $\overline{x} = 5.4$.
   Use the formula $\frac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$.

$$f'(x) = \frac{1}{x \ln 2}.$$

   (e) (7 pts) $(0, 5, 7, 7, 7)$, $\overline{x} = 5.6$.
   Use the formula $\frac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$ and the Chain Rule.

$$f'(x) = \frac{\frac{d}{dx}(3x)}{1+(3x)^2} = \frac{3}{1+9x^2}.$$

2. (8 pts) $(0, 4.8, 6.5, 8, 8)$, $\overline{x} = 5.8$.
   First, take logs of both sides:

$$\ln y = \ln \frac{(2x+1)^3}{\sqrt{x^2-1}} = 3\ln(2x+1) - \frac{1}{2}\ln(x^2-1).$$

   Then take derivatives on both sides:

$$\frac{y'}{y} = 3\left(\frac{2}{2x+1}\right) - \frac{1}{2}\left(\frac{2x}{x^2-1}\right) = \frac{6}{2x+1} - \frac{x}{x^2-1}.$$

   Finally, multiply by y and substitute the original formula:

$$y' = y\left(\frac{6}{2x+1} - \frac{x}{x^2-1}\right) = \frac{(2x+1)^3}{\sqrt{x^2-1}}\left(\frac{6}{2x+1} - \frac{x}{x^2-1}\right).$$

3. (35 pts) $(5, 18, 24, 32, 35)$, $\overline{x} = 23.6$.

(a) (7 pts) $(0, 6.8, 7, 7, 7)$, $\overline{x} = 5.7$.
Let $u = x^2 + 3x - 9$ and $du = (2x+3)dx$. Then

$$\begin{aligned} \int_2^3 \frac{2x+3}{x^2+3x-9}\,dx &= \int_1^9 \frac{du}{u} \\ &= [\ln|u|]_1^9 \\ &= \ln 9 - \ln 1 \\ &= \ln 9. \end{aligned}$$

(b) (7 pts) $(0, 3, 5, 6.3, 7)$, $\overline{x} = 4.5$.
This is a rational function in which the degree of the numerator is greater than the degree of the denominator, so first use long division to rewrite the function:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Then integrate:

$$\int \frac{x^2}{x-1}\,dx = \int \left(x + 1 + \frac{1}{x-1}\right) dx = \frac{1}{2}x^2 + x + \ln|x-1| + C.$$

(c) (7 pts) $(0, 0, 6, 7, 7)$, $\overline{x} = 4.0$.
It is best simply to memorize that

$$\int \tan x\,dx = -\ln|\cos x| + C.$$

However, it is easy enough to work out. Write $\tan x$ as $\dfrac{\sin x}{\cos x}$. Then

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

Let $u = \cos x$ and $du = -\sin x\,dx$. Then

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

(d) (7 pts) $(0, 0, 7, 7, 7)$, $\overline{x} = 4.2$.
Let $u = \ln x$ and $du = \dfrac{1}{x}\,dx$. Then

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

(e) (7 pts) $(0, 6, 7, 7, 7)$, $\overline{x} = 5.7$.

Let $u = e^x + 1$ and $du = e^x\, dx$. Then

$$\int e^x \sqrt{e^x + 1}\, dx = \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}\left(e^x + 1\right)^{3/2} + C.$$

4. (6 pts) $(0, 3.5, 5, 6, 6)$, $\overline{x} = 4.4$.

(a) Write the function as $y = \sqrt{x - 4}$ and swap x and y to get $x = \sqrt{y - 4}$. Then solve for y:

$$\begin{aligned} x &= \sqrt{y - 4} \\ x^2 &= y - 4 \\ x^2 + 4 &= y. \end{aligned}$$

So the inverse function is $f^{-1}(x) = x^2 + 4$.

(b) The domain of $f^{-1}(x)$ is the same as the range of $f(x)$ and the range of $f^{-1}(x)$ is the same as the domain of $f(x)$. The problem states that the domain of $f(x)$ is $x \geq 4$, so the range of $f^{-1}(x)$ is $y \geq 4$. It is easy to see that the range of $f(x)$ is $y \geq 0$ (positive square root), so the domain of $f^{-1}(x)$ is $x \geq 0$.

5. (6 pts) $(0, 5, 5, 6, 6)$, $\overline{x} = 4.9$.

You may use a calculator to do this, although it is easy enough without a calculator.

(a) $\arcsin 0.5 = \frac{\pi}{6}$ because $\sin \frac{\pi}{6} = 0.5$.

(b) $\sin \pi = 0$, so $\arcsin(\sin \pi) = \arcsin 0 = 0$. Be careful not to assume that $\arcsin(\sin \pi) = \pi$. Remember that the range of $\arcsin x$ is restricted in order to make it a function.

(c) $\arctan 1 = \frac{\pi}{4}$, so $\cos(\arctan 1) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

6. (10 pts) $(0, 3.5, 4, 9, 10)$, $\overline{x} = 5.6$.

(a) 2017 is 10 years after 2007, so let $t = 10$ and compute

$$P(10) = 301140000e^{(0.00896)(10)} = 301140000e^{0.0896} = 301140000(1.0937) = 329357000.$$

(b) The rate of growth of the population is the derivative of the population size. So it is

$$P'(t) = 301140000(0.00896)e^{0.00896t} = 2698214e^{0.00896t}.$$

(c) Let $t = 0$ in the formula for $P'(t)$:

$$P'(0) = 2698214e^0 = 2698214.$$

(d) Let $t = 10$ in the formula for $P'(t)$:

$$P'(10) = 2698214e^{0.0896} = 2951136.$$

(e) Solve the equation $P(t) = 2 \times 301140000$:

$$\begin{aligned} 301140000e^{0.00896t} &= 2 \times 301140000 \\ e^{0.00896t} &= 2 \\ 0.00896t &= \ln 2 \\ t &= \frac{\ln 2}{0.00896} \\ &= 77.36. \end{aligned}$$

So the population will double (according to the model) in 77.36 years, or in late 2084.