Math 142 **Test #3 Answers** **Spring 2010**

1. (24 pts)

   (a) (12 pts) Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$$\begin{aligned}\int \sin^2 x \, dx &= \frac{1}{2}\int (1 - \cos 2x)\, dx \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) \\ &= \frac{x}{2} - \frac{1}{4}\sin 2x.\end{aligned}$$

   (b) (12 pts) There are two ways to work this problem. I think the best choice is to let $u = \sec x$, $du = \tan x \sec x \, dx$. Then substitute:

$$\begin{aligned}\int \tan x \sec^4 x \, dx &= \int u^3 \, du \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}\sec^4 x + C.\end{aligned}$$

   The other possibility is to let $u = \tan x$, $du = \sec^2 x$. Then rewrite $\sec^4 x$ as $\sec^2 x \sec^2 x = (1 + \tan^2 x)\sec^2 x$ and substitute:

$$\begin{aligned}\int \tan x \sec^4 x \, dx &= \int u(1 + u^2)\, du \\ &= \int (u + u^3)\, du \\ &= \frac{1}{2}u^2 + \frac{1}{4}u^4 + C \\ &= \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.\end{aligned}$$

   The answers are equivalent.

2. (14 pts) Let $x = \sin\theta$, $dx = \cos\theta \, d\theta$, and $\sqrt{1 - x^2} = \cos\theta$. Then substitute:

$$\begin{aligned}\int_0^1 x^3\sqrt{1 - x^2} &= \int_0^{\pi/2} \sin^3 x \cos x \cos x \, dx \\ &= \int_0^{\pi/2} \sin^3 x \cos^2 x \, dx\end{aligned}$$

Now write $\sin^3 x$ as $\sin x(1-\cos^2 x)$ and use the substitution $u = \cos x$, $du = -\sin x\, dx$.

$$\begin{aligned}
\int_0^{\pi/2} \sin x^3 \cos^2 x\, dx &= \int_0^{\pi/2} \sin x(1-\cos^2 x)\cos^2 x\, dx \\
&= \int_0^{\pi/2} (\cos^2 x - \cos^4 x)\sin x\, dx \\
&= -\int_1^0 (u^2 - u^4)\, du \\
&= -\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_1^0 \\
&= -\left(0 - \left(\frac{1}{3} - \frac{1}{5}\right)\right) \\
&= \frac{2}{15}.
\end{aligned}$$

This problem can also be done (for partial credit) using integration by parts. Let $u = x^2$ and $dv = x\sqrt{1-x^2}\, dx$. Then $du = 2x\, dx$ and $v = \frac{1}{3}(1-x^2)^{3/2}$. We get

$$\begin{aligned}
\int_0^1 x^3\sqrt{1-x^2} &= \left[\frac{1}{3}x^2(1-x^2)^{3/2}\right]_0^1 - \frac{1}{3}\int_0^1 2x(1-x^2)^{3/2}\, dx \\
&= 0 - \left[\frac{1}{3}\cdot\frac{2}{5}(1-x^2)^{5/2}\right]_0^1 \\
&= -\frac{2}{15}(0-1) \\
&= \frac{2}{15}.
\end{aligned}$$

3. (26 pts)

(a) (12 pts) This function involves repeated linear factors, so write $\dfrac{x+1}{(x-2)^2}$ as $\dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$ and solve for A and B.

$$\begin{aligned}
\frac{x+1}{(x-2)^2} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} \\
x+1 &= A(x-2) + B.
\end{aligned}$$

Let $x = 2$ and get $B = 3$. Then differentiate to get $A = 1$. Now do the integration:

$$\begin{aligned}
\int \frac{x+1}{(x-2)^2}\, dx &= \int \left(\frac{1}{x-2} + \frac{3}{(x-2)^2}\right)\, dx \\
&= \ln|x-2| - \frac{3}{x-2} + C.
\end{aligned}$$

(b) (14 pts) This function involves an irreducible quadratic factor and a linear factor, so write $\dfrac{3x^2+2}{(x^2+4)(x-1)}$ as $\dfrac{Ax+B}{x^2+4}+\dfrac{C}{x-1}$ and solve for A, B, and C.

$$\begin{aligned}
\frac{3x^2+2}{(x^2+4)(x-1)} &= \frac{Ax+B}{x^2+4}+\frac{C}{x-1} \\
3x^2+2 &= (Ax+B)(x-1)+C(x^2+4) \\
&= Ax^2-Ax+Bx-B+Cx^2+4C \\
&= (A+C)x^2+(-A+B)x+(-B+4C).
\end{aligned}$$

Let $x=1$ and get $5=5C$, or $C=1$. Then the equation becomes

$$3x^2+2=(A+1)x^2+(-A+B)x+(-B+4).$$

So, $A+1=3$, $-A+B=0$, and $-B+4=2$. So $A=2$ and $B=2$. Now do the integration:

$$\begin{aligned}
\int \frac{3x^2+2}{(x^2+4)(x-1)}\,dx &= \int\left(\frac{2x+2}{x^2+4}+\frac{1}{x-1}\right)dx \\
&= \int \frac{2x}{x^2+4}\,dx+\int\frac{2}{x^2+4}\,dx+\int\frac{1}{x-1}\,dx \\
&= \ln|x^2+4|+\arctan\frac{x}{2}+\ln|x-1|+C.
\end{aligned}$$

4. (24 pts) Find the following limits, if they exist.

(a) (12 pts) This gives the indeterminate form $\frac{0}{0}$, so use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\ln(x+1)}{x^2} &= \lim_{x\to 0}\frac{1-\frac{1}{x+1}}{2x} \\
&= \lim_{x\to 0}\frac{(x+1)-1}{2x(x+1)} \\
&= \lim_{x\to 0}\frac{x}{2x(x+1)} \\
&= \lim_{x\to 0}\frac{1}{2(x+1)} \\
&= \frac{1}{2}.
\end{aligned}$$

(b) (12 pts) This gives the indeterminate form 1^∞, so first take logarithms and simplify.

$$\begin{aligned}
\ln\lim_{x\to 0}(1+x^2)^{1/x} &= \lim_{x\to 0}\ln(1+x^2)^{1/x} \\
&= \lim_{x\to 0}\frac{1}{x}\cdot\ln(1+x^2) \\
&= \lim_{x\to 0}\frac{\ln(1+x^2)}{x}.
\end{aligned}$$

This gives the indeterminate for $\frac{0}{0}$, so use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+x^2)}{x} &= \lim_{x\to 0} \frac{\left(\frac{2x}{1+x^2}\right)}{1} \\
&= \lim_{x\to 0} \frac{2x}{1+x^2} \\
&= 0.
\end{aligned}$$

Therefore, $\lim_{x\to 0} (1+x^2)^{1/x} = e^0 = 1$.

5. (12 pts) Write the improper integral as the limit of a proper integral and integrate. Then take the limit.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^{3/2}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^{3/2}}\,dx \\
&= \lim_{t\to\infty} \left[-2x^{-1/2}\right]_1^t \\
&= \lim_{t\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^t \\
&= \lim_{t\to\infty} \left(-\frac{2}{\sqrt{t}} + 2\right) \\
&= 2.
\end{aligned}$$