Mathematical Induction

Lecture 10

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Outline

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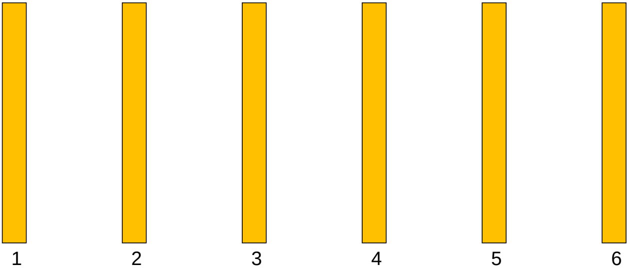
The Principle of Mathematical Induction

- Let P be a predicate about an arbitrary natural number.
- For convenience, denote it as $P(n)$, where n represents a natural number.
- To prove that $P(n)$ is true for all $n \in \mathbb{N}$, it would suffice to show that
  - $P(1)$ is true, and
  - For all natural numbers $k \geq 1$, if $P(k)$ is true, then so is $P(k+1)$.

The Principle of Mathematical Induction



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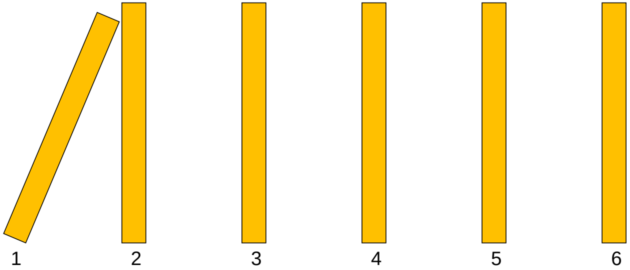

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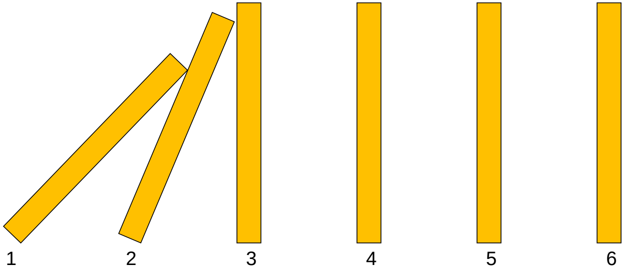

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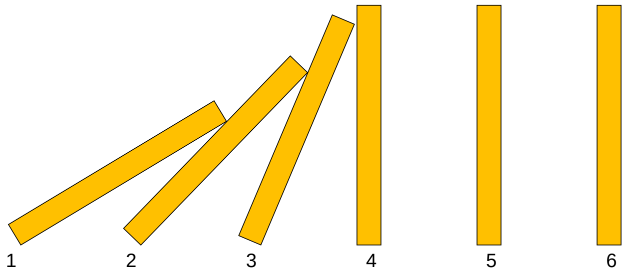

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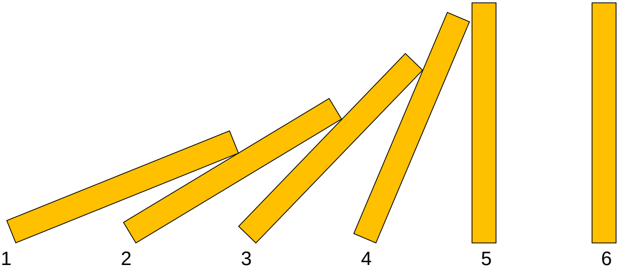

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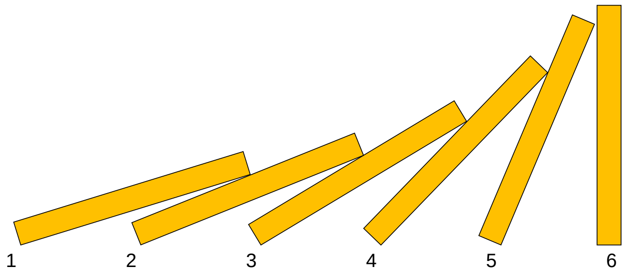

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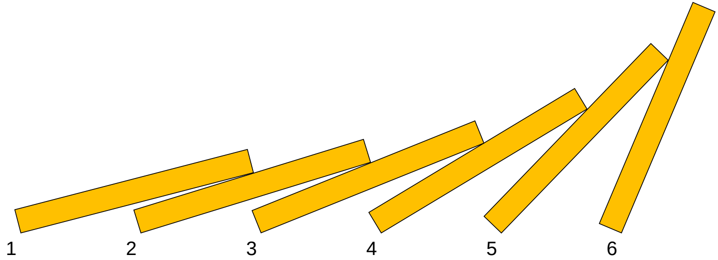

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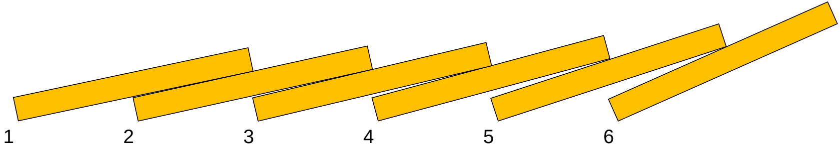

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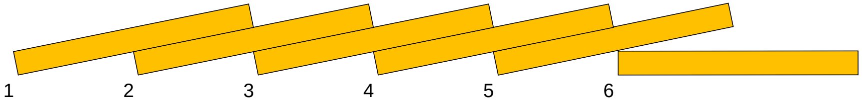

Outline

Example

Theorem

For all $n \in \mathbb{N}$, *[state the theorem, which is whatever the predicate* $P(n)$ *says.]*

Example

Proof.

Let $P(n)$ be the predicate [*write it out*].

Base step:

- When $n = 1$, [*show that* $P(1)$ *is true.*]

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then [*state what $P(k)$ says*].
- We will show that [*state what $P(k+1)$ says*]:
- (*Proceed to show that $P(k+1)$ is true. Be sure to use the "induction hypothesis" $P(k)$ at some point.*)
- Therefore, $P(k+1)$ is true.
- Thus, $P(n)$ is true for all $n \in \mathbb{N}$.

$\square$

Outline

Example

Theorem

For all $n \in \mathbb{N}$*,*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Example

Proof.

Let $P(n)$ be the predicate

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

□

Example

Proof.

Base step:

- When $n = 1$, on the left side we have

$$\sum_{i=1}^{1} i = 1$$

and on the right side we have

$$\frac{1(1+1)}{2} = 1.$$

- They are equal.
- Therefore, $P(1)$ is true.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.
- Then

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\ &= \frac{k(k+1)}{2} + k + 1\end{aligned}$$

□

Example

Proof.

Inductive step (continued):

$$
\begin{aligned}
&= (k+1)\left(\frac{k}{2}+1\right) \\
&= (k+1)\left(\frac{k+2}{2}\right) \\
&= \frac{(k+1)(k+2)}{2}.
\end{aligned}
$$

- Therefore, $P(k+1)$ is true.
- Thus, $P(n)$ is true for all $n \in \mathbb{N}$.

□

Example

Theorem

For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (3i - 2) = \frac{n(3n-1)}{2}.$$

Example

Proof.

Let $P(n)$ be the predicate

$$\sum_{i=1}^{n}(3i-2)=\frac{n(3n-1)}{2}.$$

□

Example

Proof.

Base step:

- When $n = 1$, on the left side we have

$$\sum_{i=1}^{1} (3i - 2) = 3(1) - 2 = 1$$

  and on the right side we have

$$\frac{1(3(1) - 1)}{2} = \frac{1(2)}{2} = 1.$$

- They are equal.
- Therefore, $P(1)$ is true.

$\square$

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $\sum_{i=1}^{k}(3i-2) = \frac{k(3k-1)}{2}$.
- Then

$$\begin{aligned}\sum_{i=1}^{k+1}(3i-2) &= \left(\sum_{i=1}^{k}(3i-2)\right) + (3(k+1)-2)\\ &= \frac{k(3k-1)}{2} + 3k + 1\\ &= \frac{3k^2-k}{2} + \frac{6k+2}{2}\end{aligned}$$

□

Example

Proof.

Inductive step (continued):

$$
\begin{aligned}
&= \frac{3k^2 + 5k + 2}{2} \\
&= \frac{(k+1)(3k+2)}{2} \\
&= \frac{(k+1)(3(k+1)-1)}{2}.
\end{aligned}
$$

- Therefore, $P(k+1)$ is true.
- Thus, $P(n)$ is true for all $n \in \mathbb{N}$.

□

Example

Theorem

For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.

Example

Proof.

Let $P(n)$ be the predicate $n^3 - n$ is divisible by 3.

Base step:

- When $n = 1$,
$$n^3 - n = 1^3 - 1 = 0,$$
which is divisible by 3.
- Therefore, $P(1)$ is true.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.
- So let $k^3 - k = 3m$ for some integer m.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.
- So let $k^3 - k = 3m$ for some integer m.
- We will show that $(k + 1)^3 - (k + 1)$ is also divisible by 3:

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.
- So let $k^3 - k = 3m$ for some integer m.
- We will show that $(k+1)^3 - (k+1)$ is also divisible by 3:

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.
- So let $k^3 - k = 3m$ for some integer m.
- We will show that $(k+1)^3 - (k+1)$ is also divisible by 3:

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + (3k^2 + 3k)\end{aligned}$$

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.
- So let $k^3 - k = 3m$ for some integer m.
- We will show that $(k+1)^3 - (k+1)$ is also divisible by 3:

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + (3k^2 + 3k) \\ &= 3m + 3(k^2 + k)\end{aligned}$$

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $k^3 - k$ is divisible by 3.
- So let $k^3 - k = 3m$ for some integer m.
- We will show that $(k+1)^3 - (k+1)$ is also divisible by 3:

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + (3k^2 + 3k) \\ &= 3m + 3(k^2 + k) \\ &= 3(m + k^2 + k).\end{aligned}$$

□

Example

Proof.

Inductive step (continued):

- Therefore, $P(k+1)$ is true.
- Thus, $P(n)$ is true for all $n \in \mathbb{N}$.

□

Example

Theorem

For all $n \in \mathbb{N}$, $2^n > n$.

Example

Proof.

Let $P(n)$ be the predicate $2^n > n$.

Base step:

- When $n = 1$, $2^1 = 2$ which is greater than 1.
- Therefore, $P(1)$ is true.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $2^k > k$.

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $2^k > k$.
- We will show that $2^{k+1} > k + 1$:

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $2^k > k$.
- We will show that $2^{k+1} > k + 1$:

$$2^{k+1} = 2 \cdot 2^k$$

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $2^k > k$.
- We will show that $2^{k+1} > k+1$:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k \end{aligned}$$

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $2^k > k$.
- We will show that $2^{k+1} > k + 1$:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k \\ &= k + k \end{aligned}$$

□

Example

Proof.

Inductive step:

- Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
- Then $2^k > k$.
- We will show that $2^{k+1} > k + 1$:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k \\ &= k + k \\ &\geq k + 1. \end{aligned}$$

□

Example

Proof.

Inductive step (continued):

- Therefore, $P(k+1)$ is true.
- Thus, $P(n)$ is true for all $n \in \mathbb{N}$.

□

Theorem

For all $n \in \mathbb{N}$,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \leq \frac{k}{2} + 1.$$

Outline

Definition (Recursive Sequence)

A recursive sequence is a sequence in which each term is defined as a function of previous terms and at least one term is defined nonrecursively.

First-Order Recursion

- A recursive sequence is first order if each term (except the first one) is defined in terms of one previous term.
- For example, define $\{a_n\}_{n\in\mathbb{N}}$ as

$$a_1 = 1,$$
$$a_n = 2a_{n-1} + 1 \text{ for } n \geq 2.$$

- The sequence is $1, 3, 7, 15, 31, \ldots$.

First-Order Recursion

Theorem

Define $\{a_n\}_{n \in \mathbb{N}}$ *as*

$$a_1 = 1,$$
$$a_n = 2a_{n-1} + 1 \text{ for } n \geq 2.$$

Then $a_n = 2^n - 1$ *for all* $n \in \mathbb{N}$.

First-Order Recursion

Proof.

- Let $P(n)$ be the predicate $a_n = 2^n - 1$.
- **Base step**:
  - When $n = 1$, we have $a_1 = 1$ and $2^n - 1 = 2^1 - 1 = 1$.
  - Therefore, $P(1)$ is true.

□

First-Order Recursion

Proof.

- **Inductive step**:
  - Suppose that $P(k)$ true for some $k \in \mathbb{N}$.
  - That is, suppose, for some $k \in \mathbb{N}$, that

$$a_k = 2^k - 1.$$

  - Then

$$\begin{aligned} a_{k+1} &= 2a_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= (2^{k+1} - 2) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

  - Therefore, $P(k+1)$ is true.
- Thus, $P(n)$ is true for all $n \in \mathbb{N}$.

□

Example

Example

- Define the sequence $\{a_n\}_{n \in \mathbb{N}}$ as

$$\begin{aligned} a_1 &= 2, \\ a_n &= a_{n-1} + n \cdot 2^n, \text{ for } n \in \mathbb{N}. \end{aligned}$$

- Prove that $a_n = (n-1)2^{n+1} + 2$ for $n \geq 2$.