

# The Binomial Theorem

## Lecture 11

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# Outline

- 1 The Binomial Coefficients
- 2 The Binomial Theorem
- 3 Pascal's Triangle
- 4 Binomial Random Variables

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# Permutations

## Definition (Permutation)

Given a set of  $n$  distinct objects, where  $n$  is an integer,  $n \geq 0$ , a **permutation** of those objects is an arrangement of them. (Different arrangements are different permutations.)

# Number of Permutations

## Number of Permutations

- Given  $n$  distinct objects, there are  $n!$  permutations.
- List all  $4!$  permutations of  $\{A, B, C, D\}$ .
- How many permutations are there of 25 objects?

# Combinations

## Definition (Combination)

Given a set of  $n$  distinct objects, where  $n$  is an integer,  $n \geq 0$ , and an integer  $r$ , with  $0 \leq r \leq n$ , a **combination** of those objects is a subset of that set. (Different subsets are different combinations.)

# Number of Combinations

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- Given  $n$  distinct objects and an integer  $r$ ,  $0 \leq r \leq n$ , there are

$$\frac{n!}{r!(n-r)!}$$

combinations of  $r$  objects taken from the set of  $n$  objects.

- This number is denoted  $\binom{n}{r}$ .
- List all of the combinations size 2 of  $\{A, B, C, D\}$ .
- How many combinations are there of 25 objects taken 5 at a time?

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# The Binomial Theorem

## Theorem (The Binomial Theorem)

Let  $a, b \in \mathbb{R}$  and let  $n$  and  $r$  be integers with  $n \geq 0$  and  $0 \leq r \leq n$ . In the expansion of  $(a + b)^n$ , the coefficient of  $a^r b^{n-r}$  is  $\binom{n}{r}$ . That is,

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} b^n.$$

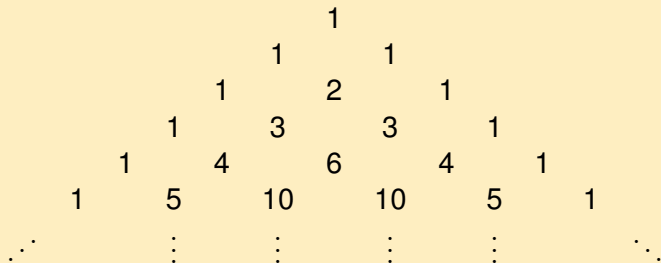
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# Pascal's Triangle

## Pascal's Triangle

The binomial coefficients may be arranged in a triangle, called **Pascal's Triangle**.



# Pascal's Formula

## Theorem (Pascal's Formula)

Let  $n$  and  $r$  be integers with  $n \geq 0$  and  $0 \leq r \leq n$ . Then

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$

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  - Those subsets that contain  $x$ .
  - Those subsets that do not contain  $x$ .



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- Divide the subsets of  $S$  of  $r$  elements into two categories:
  - Those subsets that contain  $x$ .
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- Remove  $x$  from each subset in the first group and it is clear that the first group contains  $\binom{n-1}{r-1}$  subsets.



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- Divide the subsets of  $S$  of  $r$  elements into two categories:
  - Those subsets that contain  $x$ .
  - Those subsets that do not contain  $x$ .
- Remove  $x$  from each subset in the first group and it is clear that the first group contains  $\binom{n-1}{r-1}$  subsets.
- It is also clear that there are  $\binom{n-1}{r}$  subsets in the second group.
- Therefore,  $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$ .





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- The trials are independent.

## Definition (Binomial Random Variable)

A **binomial random variable** is a random variable whose value is the number of successes in a binomial experiment.

# Binomial Random Variables

## Theorem (The Binomial Distribution)

*Let  $X$  be a binomial random variable with  $n$  trials,  $n \geq 0$ , and probability  $p$  of success. Let  $r$  be an integer with  $0 \leq r \leq n$ . Then the probability of exactly  $r$  successes is*

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}.$$