The Binomial Theorem

Lecture 11

Robb T. Koether

Hampden-Sydney College

Mon, Mar 27, 2017

Outline

1. The Binomial Coefficients
2. The Binomial Theorem
3. Pascal's Triangle
4. Binomial Random Variables

Outline

Permutations

Definition (Permutation)

Given a set of n distinct objects, where n is an integer, $n \geq 0$, a permutation of those objects is an arrangement of them. (Different arrangements are different permutations.)

Number of Permutations

Number of Permutations

- Given n distinct objects, there are $n!$ permutations.
- List all 4! permutations of $\{A, B, C, D\}$.
- How many permutations are there of 25 objects?

Combinations

Definition (Combination)

Given a set of n distinct objects, where n is an integer, $n \geq 0$, and an integer r, with $0 \leq r \leq n$, a combination of those objects is a subset of that set. (Different subsets are different combinations.)

Number of Combinations

Number of Combinations

- Given n distinct objects and an integer r, $0 \le r \le n$, there are

$$\frac{n!}{r!(n-r)!}$$

  combinations of r objects taken from the set of n objects.
- This number is denoted $\binom{n}{r}$.
- List all of the combinations size 2 of $\{A, B, C, D\}$.
- How many combinations are there of 25 objects taken 5 at a time?

Outline

The Binomial Theorem

Theorem (The Binomial Theorem)

Let $a, b \in \mathbb{R}$ and let n and r be an integers with $n \geq 0$ and $0 \leq r \leq n$. In the expansion of $(a+b)^n$, the coefficient of $a^r b^{n-r}$ is $\binom{n}{r}$. That is,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n.$$

Outline

Pascal's Triangle

Pascal's Triangle

The binomial coefficients may be arranged in a triangle, called Pascal's Triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\iddots & & & \vdots & & \vdots & & \vdots & & \vdots & & & \ddots
\end{array}
$$

Pascal's Formula

Theorem (Pascal's Formula)

Let n and r be integers with $n \geq 0$ and $0 \leq r \leq n$. Then

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$

Number of Combinations

Proof.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)
- Divide the subsets of S of r elements into two categories:

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)
- Divide the subsets of S of r elements into two categories:
  - Those subsets that contain x.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)
- Divide the subsets of S of r elements into two categories:
  - Those subsets that contain x.
  - Those subsets that do not contain x.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)
- Divide the subsets of S of r elements into two categories:
  - Those subsets that contain x.
  - Those subsets that do not contain x.
- Remove x from each subset in the first group and it is clear that the first group contains $\binom{n-1}{r-1}$ subsets.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)
- Divide the subsets of S of r elements into two categories:
  - Those subsets that contain x.
  - Those subsets that do not contain x.
- Remove x from each subset in the first group and it is clear that the first group contains $\binom{n-1}{r-1}$ subsets.
- It is also clear that there are $\binom{n-1}{r}$ subsets in the second group.

□

Number of Combinations

Proof.

- The base cases of $n = 0$ and $n = 1$ are obvious.
- Suppose that the predicate is true for some $k \geq 1$.
- Consider a set S of $k + 1$ elements.
- Select an arbitrary element $x \in S$.
- Let r be an integer $1 \leq r \leq k + 1$. (Handle $r = 0$ separately.)
- Divide the subsets of S of r elements into two categories:
  - Those subsets that contain x.
  - Those subsets that do not contain x.
- Remove x from each subset in the first group and it is clear that the first group contains $\binom{n-1}{r-1}$ subsets.
- It is also clear that there are $\binom{n-1}{r}$ subsets in the second group.
- Therefore, $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$.

□

Outline

Binomial Random Variables

Definition (Binomial Experiment)

A binomial experiment is a procedure with the follow characteristics.

Binomial Random Variables

Definition (Binomial Experiment)

A binomial experiment is a procedure with the follow characteristics.

- It consists of n trials, for $n \geq 0$.

Binomial Random Variables

Definition (Binomial Experiment)

A binomial experiment is a procedure with the follow characteristics.

- It consists of n trials, for $n \geq 0$.
- Each trial ends in one of exactly two possible outcomes, denoted "success" and "failure."

Binomial Random Variables

Definition (Binomial Experiment)

A binomial experiment is a procedure with the follow characteristics.

- It consists of n trials, for $n \geq 0$.
- Each trial ends in one of exactly two possible outcomes, denoted "success" and "failure."
- The probability of success p is the same on each trial.

Binomial Random Variables

Definition (Binomial Experiment)

A binomial experiment is a procedure with the follow characteristics.

- It consists of n trials, for $n \geq 0$.
- Each trial ends in one of exactly two possible outcomes, denoted "success" and "failure."
- The probability of success p is the same on each trial.
- The trials are independent.

Binomial Random Variables

Definition (Binomial Experiment)

A binomial experiment is a procedure with the follow characteristics.

- It consists of n trials, for $n \geq 0$.
- Each trial ends in one of exactly two possible outcomes, denoted "success" and "failure."
- The probability of success p is the same on each trial.
- The trials are independent.

Definition (Binomial Random Variable)

A binomial random variable is a random variable whose value is the number of successes in a binomial experiment.

Binomial Random Variables

Theorem (The Binomial Distribution)

Let X be a binomial random variable with n trials, $n \geq 0$, and probability p of success. Let r be an integer with $0 \leq r \leq n$. Then the probability of exactly r successes is

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}.$$