Relations

Lecture 12

Robb T. Koether

Hampden-Sydney College

Mon, Apr 3, 2017

Outline

Outline

Cartesian Product

Definition (Cartesian Product)

Given two sets A and B, the Cartesian product $A \times B$ is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

- The definition can be extended to any number of sets:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i\}.$$

Relations

Definition (Relation)

Given two sets A and B, a relation from A to B is a subset of $A \times B$. If A and B are the same set, then we say that the relation is on the set A.

- The relation is the set of all the pairs of elements (a, b) from $A \times B$ such that a has the relationship to b.

Notation

- If we give the relation a name, for example R, then we might write

$$aRb$$

  to mean $(a, b) \in R$.
- We often use the generic symbol $\sim$ and write $a \sim b$.

Example

Example (Relation)

- Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10\}$.
- Let the relation $R \subseteq A \times B$ be the set of ordered pairs (a, b) such that $a \mid b$.
- List the elements of the relation R.

Outline

The Reflexive Property

Definition (The Reflexive Property)

A relation $\sim$ on a set A is reflexive if, for all $a \in A$,

$$a \sim a.$$

The Reflexive Property

Definition (The Reflexive Property)

A relation $\sim$ on a set A is reflexive if, for all $a \in A$,

$$a \sim a.$$

- The divisibility relation $|$ is reflexive on $\mathbb{N}$, but it is not reflexive on $\mathbb{Z}$.

The Reflexive Property

Definition (The Reflexive Property)

A relation $\sim$ on a set A is reflexive if, for all $a \in A$,

$$a \sim a.$$

- The divisibility relation $\mid$ is reflexive on $\mathbb{N}$, but it is not reflexive on $\mathbb{Z}$.
- The relation $\geq$ is reflexive on $\mathbb{R}$.

The Reflexive Property

Definition (The Reflexive Property)

A relation $\sim$ on a set A is reflexive if, for all $a \in A$,

$$a \sim a.$$

- The divisibility relation $|$ is reflexive on $\mathbb{N}$, but it is not reflexive on $\mathbb{Z}$.
- The relation $\geq$ is reflexive on $\mathbb{R}$.
- The relation $\sim$ on $\mathbb{R}$, defined by $a \sim b$ if $|a - b| < 1$, is reflexive.

The Symmetric Property

Definition (The Symmetric Property)

A relation $\sim$ on a set A is symmetric if, for all $a, b \in A$,

$$a \sim b \Longrightarrow b \sim a.$$

The Symmetric Property

Definition (The Symmetric Property)

A relation $\sim$ on a set A is symmetric if, for all $a, b \in A$,

$$a \sim b \Longrightarrow b \sim a.$$

- The divisibility relation is not symmetric on $\mathbb{N}$.

The Symmetric Property

Definition (The Symmetric Property)

A relation $\sim$ on a set A is symmetric if, for all $a, b \in A$,

$$a \sim b \Longrightarrow b \sim a.$$

- The divisibility relation is not symmetric on $\mathbb{N}$.
- The relation $\geq$ is not symmetric on $\mathbb{R}$.

The Symmetric Property

Definition (The Symmetric Property)

A relation $\sim$ on a set A is symmetric if, for all $a, b \in A$,

$$a \sim b \Longrightarrow b \sim a.$$

- The divisibility relation is not symmetric on $\mathbb{N}$.
- The relation $\geq$ is not symmetric on $\mathbb{R}$.
- The relation $\sim$ on $\mathbb{R}$, defined by $a \sim b$ if $|a - b| < 1$, is symmetric.

The Transitive Property

Definition (The Transitive Property)

A relation $\sim$ on a set A is transitive if, for all $a, b, c \in A$,

$$(a \sim b \text{ and } b \sim c) \Longrightarrow a \sim c.$$

The Transitive Property

Definition (The Transitive Property)

A relation $\sim$ on a set A is transitive if, for all $a, b, c \in A$,

$$(a \sim b \text{ and } b \sim c) \implies a \sim c.$$

- The divisibility relation is transitive on $\mathbb{N}$ and on $\mathbb{Z}$.

The Transitive Property

Definition (The Transitive Property)

A relation $\sim$ on a set A is transitive if, for all $a, b, c \in A$,

$$(a \sim b \text{ and } b \sim c) \implies a \sim c.$$

- The divisibility relation is transitive on $\mathbb{N}$ and on $\mathbb{Z}$.
- The relation $\geq$ is transitive on $\mathbb{R}$.

The Transitive Property

Definition (The Transitive Property)

A relation $\sim$ on a set A is transitive if, for all $a, b, c \in A$,

$$(a \sim b \text{ and } b \sim c) \implies a \sim c.$$

- The divisibility relation is transitive on $\mathbb{N}$ and on $\mathbb{Z}$.
- The relation $\geq$ is transitive on $\mathbb{R}$.
- The relation $\sim$ on $\mathbb{R}$, defined by $a \sim b$ if $|a - b| < 1$, is not transitive.

The Antisymmetric Property

Definition (The Antisymmetric Property)

A relation $\sim$ on a set A is antisymmetric if, for all $a, b \in A$,

$$(a \sim b \text{ and } b \sim a) \implies a = b.$$

The Antisymmetric Property

Definition (The Antisymmetric Property)

A relation $\sim$ on a set A is antisymmetric if, for all $a, b \in A$,

$$(a \sim b \text{ and } b \sim a) \implies a = b.$$

- The divisibility relation is antisymmetric on $\mathbb{N}$, but not on $\mathbb{Z}$.

The Antisymmetric Property

Definition (The Antisymmetric Property)

A relation $\sim$ on a set A is antisymmetric if, for all $a, b \in A$,

$$(a \sim b \text{ and } b \sim a) \implies a = b.$$

- The divisibility relation is antisymmetric on $\mathbb{N}$, but not on $\mathbb{Z}$.
- The relation $\geq$ is antisymmetric on $\mathbb{R}$.

The Antisymmetric Property

Definition (The Antisymmetric Property)

A relation $\sim$ on a set A is antisymmetric if, for all $a, b \in A$,

$$(a \sim b \text{ and } b \sim a) \implies a = b.$$

- The divisibility relation is antisymmetric on $\mathbb{N}$, but not on $\mathbb{Z}$.
- The relation $\geq$ is antisymmetric on $\mathbb{R}$.
- The relation $\sim$ on $\mathbb{R}$, defined by $a \sim b$ if $|a - b| < 1$, is not antisymmetric.

Symmetry and Antisymmetry

Theorem

If a relation $\sim$ on a set A is reflexive, symmetric, and antisymmetric, then it is the equality, or identity, relation.

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.

$\square$

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.
- By reflexivity, every element has the relation to itself.

$\square$

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.
- By reflexivity, every element has the relation to itself.
- We will show that every element can have the relation *only* to itself, and thus it must be the identity relation.

□

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.
- By reflexivity, every element has the relation to itself.
- We will show that every element can have the relation *only* to itself, and thus it must be the identity relation.
- Let $a, b \in A$ and suppose that $a \sim b$. We will show that $a = b$.

□

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.
- By reflexivity, every element has the relation to itself.
- We will show that every element can have the relation *only* to itself, and thus it must be the identity relation.
- Let $a, b \in A$ and suppose that $a \sim b$. We will show that $a = b$.
- Then, by symmetry, we must also have $b \sim a$.

□

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.
- By reflexivity, every element has the relation to itself.
- We will show that every element can have the relation *only* to itself, and thus it must be the identity relation.
- Let $a, b \in A$ and suppose that $a \sim b$. We will show that $a = b$.
- Then, by symmetry, we must also have $b \sim a$.
- So, by antisymmetry, it follows that $a = b$.

□

Symmetry and Antisymmetry

Proof.

- Let $\sim$ be a reflexive, symmetric, and antisymmetric relation on a set A.
- By reflexivity, every element has the relation to itself.
- We will show that every element can have the relation *only* to itself, and thus it must be the identity relation.
- Let $a, b \in A$ and suppose that $a \sim b$. We will show that $a = b$.
- Then, by symmetry, we must also have $b \sim a$.
- So, by antisymmetry, it follows that $a = b$.
- Therefore, $\sim$ is the equality (or identity) relation.

□

A Bogus Theorem

Theorem (Bogus Theorem)

If a relation $\sim$ on a set A is symmetric and transitive, then it is also reflexive.

A Bogus Proof

Proof.

- Let $\sim$ be a symmetric and transitive relation on the set A.

$\square$

A Bogus Proof

Proof.

- Let $\sim$ be a symmetric and transitive relation on the set A.
- Let $a \in A$. We will show that $a \sim a$.

$\square$

A Bogus Proof

Proof.

- Let $\sim$ be a symmetric and transitive relation on the set A.
- Let $a \in A$. We will show that $a \sim a$.
- Let b be any element in A such that $a \sim b$.

□

A Bogus Proof

Proof.

- Let $\sim$ be a symmetric and transitive relation on the set A.
- Let $a \in A$. We will show that $a \sim a$.
- Let b be any element in A such that $a \sim b$.
- Then, by symmetry, $b \sim a$.

$\square$

A Bogus Proof

Proof.

- Let $\sim$ be a symmetric and transitive relation on the set A.
- Let $a \in A$. We will show that $a \sim a$.
- Let b be any element in A such that $a \sim b$.
- Then, by symmetry, $b \sim a$.
- By transitivity, because $a \sim b$ and $b \sim a$, it follows that $a \sim a$.

□

A Bogus Proof

Proof.

- Let $\sim$ be a symmetric and transitive relation on the set A.
- Let $a \in A$. We will show that $a \sim a$.
- Let b be any element in A such that $a \sim b$.
- Then, by symmetry, $b \sim a$.
- By transitivity, because $a \sim b$ and $b \sim a$, it follows that $a \sim a$.
- Therefore, $\sim$ is reflexive.

□

A Bogus Proof

- Find the error in the proof.
- Then find an example of a set A and a relation $\sim$ on A that is symmetric and transitive, but is not reflexive.

Outline

Equivalence Relations

Definition

An equivalence relation on a set A is a relation $\sim$ on A that is reflexive, symmetric, and transitive.

Examples

Example (Equivalence Relations)

- Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $|x| = |y|$.
- Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x^2 = y^2$.
- Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $\cos x = \cos y$.
- Let $d \in \mathbb{N}$ and define $\sim$ on $\mathbb{Z}$ by $a \sim b$ if $d \mid (a - b)$.
- Let S be a finite set and define $\sim$ on $\mathcal{P}(S)$ by $A \sim B$ if A and B have the same number of elements.