Partial Order Relations

Lecture 13

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Outline

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1. Equivalence Relations
2. Equivalence Classes
3. Partial Order Relations
4. Greatest Lower Bound and Least Upper Bound

Equivalence Relations

Definition

An equivalence relation on a set A is a relation $\sim$ on A that is reflexive, symmetric, and transitive.

Examples

Example (Equivalence Relations)

- Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $|x| = |y|$.
- Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x^2 = y^2$.
- Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $\cos x = \cos y$.
- Let $d \in \mathbb{N}$ and define $\sim$ on $\mathbb{Z}$ by $a \sim b$ if $d \mid (a - b)$.
- Let S be a finite set and define $\sim$ on $\mathcal{P}(S)$ by $A \sim B$ if A and B have the same number of elements.

Outline

Equivalence Classes

Definition (Equivalence Classes)

Let A be a set with equivalence relation $\sim$ and let $a \in A$. The equivalence class of a is the set

$$[a] = \{x \in A \mid x \sim a\}.$$

Examples

Example (Equivalence Classes)

In the previous examples of equivalence relations, describe the equivalence classes.

Outline

Partial Order Relations

Definition (Partial Order Relation)

A relation $\preceq$ on a set A is a partial order relation if $\preceq$ is reflexive, antisymmetric, and transitive.

Example

Example (Partial Order Relations)

- For any set A, the subset relation $\subseteq$ is a partial order relation on $\mathcal{P}(A)$.
- The divisibility relation $|$ is a partial order on $\mathbb{N}$.
- The relation $\preceq$ on $\mathbb{R} \times \mathbb{R}$ defined by

$$(a, b) \preceq (c, d) \text{ if } a \leq c \text{ and } b \leq d$$

is a partial order relation.

Example

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- For any set A, the subset relation $\subseteq$ is a partial order relation on $\mathcal{P}(A)$.
- The divisibility relation $|$ is a partial order on $\mathbb{N}$.
- The relation $\preceq$ on $\mathbb{R} \times \mathbb{R}$ defined by

$$(a, b) \preceq (c, d) \text{ if } a \leq c \text{ and } b \leq d$$

  is a partial order relation.
  - If "and" were replaced by "or," would this still be a partial order relation?

Diagrams

- A partial order relation on a finite set may be represented as a diagram. The relation $a \preceq b$ is drawn as
  - a and b are connected by a line.
  - a is drawn below b.
- The result is called a Hasse diagram.

Diagrams

Example (Diagrams)

- Draw a diagram for the subset relation where $A = \{1, 2, 3\}$.
- Draw a diagram for the divisibility relation where $A = \{1, 2, 3, 4, 6, 12\}$.
- Draw a diagram for the divisibility relation where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Diagrams

Example (Diagrams)

- For the relation defined earlier on $\mathbb{R} \times \mathbb{R}$, shade the part of the plane that contains all points (x, y)
  - Such that $(1, 2) \preceq (x, y)$.
  - Such that $(x, y) \preceq (1, 2)$.

Outline

Greatest Lower Bound

Definition (Greatest Lower Bound)

Let $\preceq$ be a partial order on a set A and let $a, b \in A$. The greatest lower bound of a and b, denoted $\text{glb}(a, b)$, is an element $g \in A$ such that

1. $g \preceq a$ and $g \preceq b$, and
2. For all $c \in A$, if $c \preceq a$ and $c \preceq b$, then $c \preceq g$.

Least Upper Bound

Definition (Least Upper Bound)

Let $\preceq$ be a partial order on a set A and let $a, b \in A$. The least upper bound of a and b, denoted $\text{lub}(a, b)$, is an element $\ell \in A$ such that

1. $a \preceq \ell$ and $b \preceq \ell$, and
2. For all $c \in A$, if $a \preceq c$ and $b \preceq c$, then $\ell \preceq d$.

Examples

- In the previous examples, find the greatest lower bound and least upper bound of some pairs of elements.
- Identify the glb and the lub in the diagrams.

Example

Theorem

Let A be a set with partial order $\preceq$. Define a relation $a \succeq b$ on A to mean $b \preceq a$. ($\succeq$ is the "reverse" of $\preceq$.) Then $\succeq$ is also a partial order relation.

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.

$\square$

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$.

□

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$.

$\square$

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.

$\square$

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$.

□

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$. Then $b \preceq a$ and $a \preceq b$, which implies that $a = b$.

□

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$. Then $b \preceq a$ and $a \preceq b$, which implies that $a = b$. Therefore, $\succeq$ is antisymmetric.

□

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$. Then $b \preceq a$ and $a \preceq b$, which implies that $a = b$. Therefore, $\succeq$ is antisymmetric.
- **Transitive:** Let $a, b, c \in A$ and suppose that $a \succeq b$ and $b \succeq c$.

□

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$. Then $b \preceq a$ and $a \preceq b$, which implies that $a = b$. Therefore, $\succeq$ is antisymmetric.
- **Transitive:** Let $a, b, c \in A$ and suppose that $a \succeq b$ and $b \succeq c$. Then $b \preceq a$ and $c \preceq b$, so $c \preceq a$.

$\square$

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$. Then $b \preceq a$ and $a \preceq b$, which implies that $a = b$. Therefore, $\succeq$ is antisymmetric.
- **Transitive:** Let $a, b, c \in A$ and suppose that $a \succeq b$ and $b \succeq c$. Then $b \preceq a$ and $c \preceq b$, so $c \preceq a$. That implies that $a \succeq c$.

□

Example

Proof.

- Let A be a set with partial order $\preceq$ and define $a \succeq b$ to mean $b \preceq a$.
- **Reflexive:** Let $a \in A$. Then $a \preceq a$, which implies that $a \succeq a$. Therefore, $\succeq$ is reflexive.
- **Antisymmetric:** Let $a, b \in A$ and suppose that $a \succeq b$ and $b \succeq a$. Then $b \preceq a$ and $a \preceq b$, which implies that $a = b$. Therefore, $\succeq$ is antisymmetric.
- **Transitive:** Let $a, b, c \in A$ and suppose that $a \succeq b$ and $b \succeq c$. Then $b \preceq a$ and $c \preceq b$, so $c \preceq a$. That implies that $a \succeq c$. Therefore, $\succeq$ is transitive.

$\square$