Functions

Lecture 14

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Outline

1. Functions
2. Properties of Functions
3. Inverse Functions
4. Composition of Functions
5. Images and Preimages

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Functions

Definition (Function)

Given two sets X and Y, a function $f : X \to Y$ is a relation from X to Y with the following two properties.

(1) For every $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$, and

(2) For every $x \in X$, if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

- We traditionally write $f(x) = y$ instead of $(x, y) \in f$, but they mean the same thing.

Domain and Codomain

Definition (Domain and Codomain)

Let $f : X \to Y$ be a function. The set X is the domain of f and the set Y is the codomain of f.

Examples

Example

- Find a relation from $\mathbb{R}$ to $\mathbb{R}$ that violates properties (1) and (2).

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- Find a relation from $\mathbb{R}$ to $\mathbb{R}$ that violates property (2), but not (1).
- Find a relation from $\mathbb{R}$ to $\mathbb{R}$ that satisfies properties (1) and (2).

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One-to-one Functions

Definition (One-to-one)

A function $f : X \to Y$ is one-to-one if for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

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- Compare that to

  (2) For every $x \in X$, if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

Examples

Example (One-to-one Functions)

Which of the following functions are one-to-one?

- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

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- $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = x + \ln x$.
- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + \sin x$.

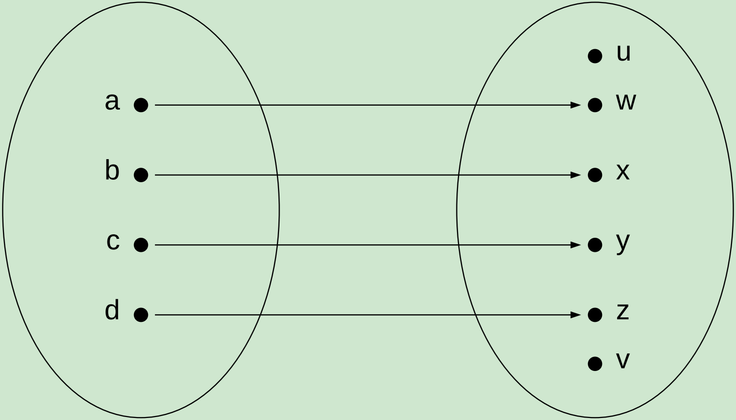
One-to-one Correspondence

Definition (Onto)

A function $f : X \to Y$ is a one-to-one correspondence if it is one-to-one and onto.

One-to-one and Onto

Example (One-to-one and Onto)



Is this function one-to-one? Onto?

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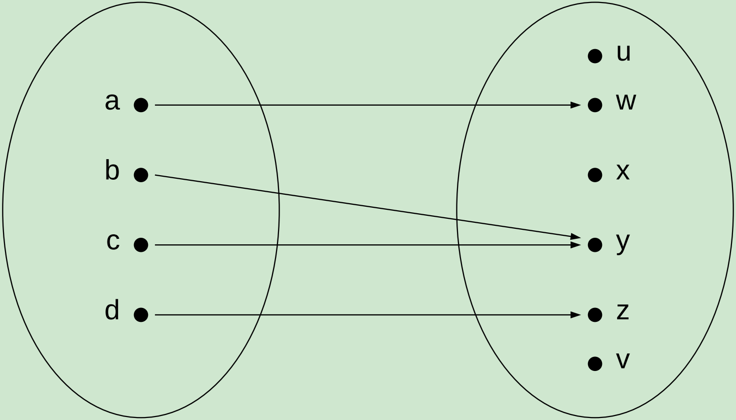

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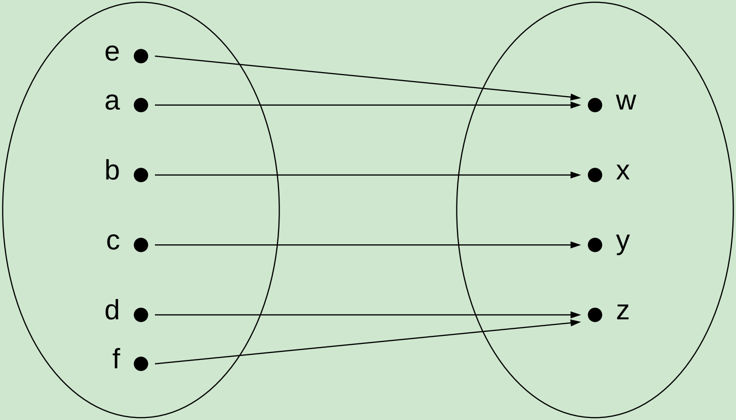

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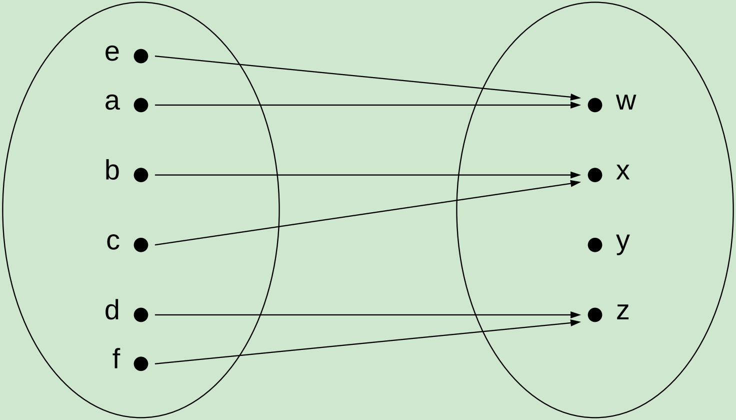

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Inverse Relations

Definition (Inverse Relation)

Let R be a relation from a set A to a set B. The inverse relation R^{-1} is from B to A and is defined as

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Inverse Functions

- The inverse of a function is a relation, but is not necessarily a function.
- For f^{-1} to be a function, it must satisfy the two properties.
  1. For all $y \in Y$, there exists $x \in X$ such that $(y, x) \in f^{-1}$.
  2. For every $y \in Y$, if $(y, x_1) \in f^{-1}$ and $(y, x_2) \in f^{-1}$, then $x_1 = x_2$.
- That is, f^{-1} is a function mapping Y to X if $f\colon X \to Y$ is one-to-one and onto.

Examples

Example (Onto Functions)

For which of the following functions is f^{-1} a function?

- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

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- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.
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For which of the following functions is f^{-1} a function?

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- $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = x^2$.
- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.
- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x+3}{5}$.

Examples

Example (Onto Functions)

For which of the following functions is f^{-1} a function?

- $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.
- $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = x^2$.
- $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.
- $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x+3}{5}$.
- $f\colon \mathbb{R} \to \mathbb{R}^+$ by $f(x) = e^x - e^{-x}$.

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For which of the following functions is f^{-1} a function?

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- $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x+3}{5}$.
- $f: \mathbb{R} \to \mathbb{R}^+$ by $f(x) = e^x - e^{-x}$.
- $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$.

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- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.
- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x+3}{5}$.
- $f : \mathbb{R} \to \mathbb{R}^+$ by $f(x) = e^x - e^{-x}$.
- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$.
- $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ by $f(x) = \sin x$.

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Composition of Functions

Definition (Composition of Functions)

Let X, Y, and Z be sets and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. The composition of f and g, denoted $g \circ f\colon X \to Z$, is defined by $(f \circ g)(x) = g(f(x))$.

Composition

Example

- Let $Y = \{x \in \mathbb{R} \mid x \geq 0\}$.
- In each case, describe $f \circ g$ and $g \circ f$.
  - Let $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x+1}{3}$, $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = \frac{3-x}{2}$.
  - Let $f\colon \mathbb{R} \to Y$ by $f(x) = x^2$, $g\colon Y \to \mathbb{R}$ by $g(x) = \sqrt{x}$.
  - Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \ln x$, $g\colon \mathbb{R} \to \mathbb{R}^+$ by $g(x) = e^{2x}$.

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Image

Definition (Image)

For sets X and Y, let $f\colon X \to Y$ and let $A \subseteq X$. The image of A is the set

$$f[A] = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

- We could also say that $f[A] = \{f(x) \in Y \mid x \in A\}$.

Preimage

Definition (Preimage)

For sets X and Y, let $f\colon X \to Y$ and let $B \subseteq Y$. The preimage of B is the set

$$f^{-1}[B] = \{x \in X \mid f(x) \in B\}.$$

Image and Preimage

- If $f\colon X \to Y$ and $A \subseteq X$, what is $f^{-1}[f[A]]$?
- If $B \subseteq Y$, what is $f[f^{-1}[B]]$?
- If f is not one-to-one or onto, what can we say about $f^{-1}[f[A]]$ and $f[f^{-1}[B]]$?