Cardinality
Lecture 15

Robb T. Koether

Hampden-Sydney College

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Outline

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- Each morning as the sheep were put out to pasture, they passed through a gate.
- As each sheep passed through the gate, the little shepherd boy would place a pebble in a sack.
- At the end of the day, the sheep were brought in from the pasture for the night.

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- Each morning as the sheep were put out to pasture, they passed through a gate.
- As each sheep passed through the gate, the little shepherd boy would place a pebble in a sack.
- At the end of the day, the sheep were brought in from the pasture for the night.
- Again, each sheep passed through the gate.

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- After the last sheep had passed through, if the sack was empty, then all the sheep were accounted for.
- But if the sack was not empty, then the little shepherd boy knew that he was missing as many sheep as he had pebbles left in the sack and he would go looking for the lost sheep.
- And that is how the little shepherd boy used calculus to count his sheep.

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- If the two sets were in one-to-one correspondence, then they must have the same number of elements.
- So, how did that use calculus?
- “Calculus” is Latin for “pebble.”
- *calx* (“limestone”) + *-ulus* (diminutive suffix)

Outline

Cardinality

Definition (Cardinality)

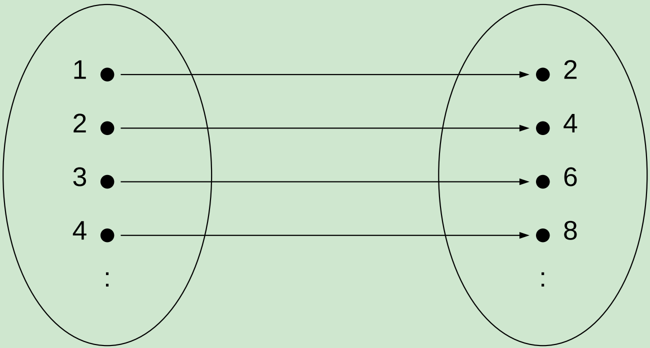
Let A and B be sets. Then A and B have the same cardinality, denoted $|A| = |B|$, if there exists a one-to-one correspondence $f: A \to B$. Otherwise, they have unequal cardinalities.

- Note that we did not define "cardinality," but only "the same cardinality" and "unequal cardinality."

Example

Example (The Same Cardinality)

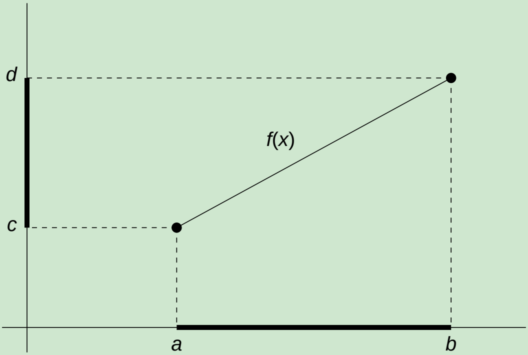
Let $2\mathbb{N} = \{2n \mid n \in \mathbb{N}\}$. Prove that $\mathbb{N}$ and $2\mathbb{N}$ have the same cardinality.



Example

Example (The Same Cardinality)

Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Prove that the open intervals (a, b) and (c, d) have the same cardinality.



Example

Example (The Same Cardinality)

Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Prove that (a, b) and (c, d) have the same cardinality.

Countably Infinite and Uncountable

Definition (Finite)

A set A is finite if $|A| = |\{1, 2, 3, \ldots, n\}|$ for some $n \in \mathbb{N}$.

Definition (Countably Infinite)

A set A is countably infinite if $|A| = |\mathbb{N}|$.

Definition (Uncountable)

A set A is uncountable if A is not finite and if A is not countably infinite.

Listing Elements

- If a set is finite, then its elements can be listed (obviously).
- If a set if countably infinite, then its elements be listed.
  - Let A be a countably infinite set and let $f : \mathbb{N} \to A$ be a one-to-one correspondence.
  - Then $f(1), f(2), f(3), \ldots$ is a listing of the elements of A.

Listing Elements

- We may prove that a set is countably infinite by describing a method of listing its elements that meets the following criteria:
  - Every element of the set occurs at a finite position in the list (onto).
  - No element occurs more than once in the list (one-to-one).

Listing Elements

- Let A be the set of all finite sequences of 0s and 1s.
- Prove that A is countably infinite.
- False method:
  - List all the sequences that begin with 0, then list all the sequences that begin with 1.
- What is wrong with that?

Outline

$\mathbb{Q}$ is Countable

Theorem

$\mathbb{Q}$ *is countable.*

$\mathbb{N} \times \mathbb{N}$ is Countable

Lemma

$\mathbb{N} \times \mathbb{N}$ *is countable.*

$\mathbb{N} \times \mathbb{N}$ is Countable

Proof.

- We may represent $\mathbb{N} \times \mathbb{N}$ as a table infinite to the right and down.

	1	2	3	$\cdots$	n	$\cdots$
1	$(1,1)$	$(1,2)$	$(1,3)$	$\cdots$	$(1,n)$	$\cdots$
2	$(2,1)$	$(2,2)$	$(2,3)$	$\cdots$	$(2,n)$	$\cdots$
3	$(3,1)$	$(3,2)$	$(3,3)$	$\cdots$	$(3,n)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
n	$(n,1)$	$(n,2)$	$(n,3)$	$\cdots$	(n,n)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

□

$\mathbb{N} \times \mathbb{N}$ is Countable

Proof.

- List the pairs in order by traversing the diagonals from upper right to lower left.

□

$\mathbb{N} \times \mathbb{N}$ is Countable

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- List the pairs in order by traversing the diagonals from upper right to lower left.
- The list begins $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots$.

$\square$

$\mathbb{N} \times \mathbb{N}$ is Countable

Proof.

- List the pairs in order by traversing the diagonals from upper right to lower left.
- The list begins $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots$.
- An equivalent description of the list is
  - Arrange the pairs (m, n) in order of increasing sums $m + n$.
  - Within groups, arrange the pairs (m, n) in order of increasing m.

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$\mathbb{N} \times \mathbb{N}$ is Countable

Proof.

- List the pairs in order by traversing the diagonals from upper right to lower left.
- The list begins $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots$.
- An equivalent description of the list is
  - Arrange the pairs (m, n) in order of increasing sums $m + n$.
  - Within groups, arrange the pairs (m, n) in order of increasing m.
- Thus, $\mathbb{N} \times \mathbb{N}$ is countable.

$\square$

$\mathbb{Q}$ is Countable

Proof that $\mathbb{Q}$ is countable.

- We may represent $\mathbb{Q}$ as a table infinite to the right and down.
- The row represents the numerator and the column represents the denominator.

	1	2	3	$\cdots$	n	$\cdots$
1	$1/1$	$1/2$	$1/3$	$\cdots$	$1/n$	$\cdots$
2	$2/1$	$2/2$	$2/3$	$\cdots$	$2/n$	$\cdots$
3	$3/1$	$3/2$	$3/3$	$\cdots$	$3/n$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
n	$n/1$	$n/2$	$n/3$	$\cdots$	n/n	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

□

$\mathbb{Q}$ is Countable

Proof.

- List the rationals in order by traversing the table as before, but skip over the unreduced rationals.

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- List the rationals in order by traversing the table as before, but skip over the unreduced rationals.
- The list begins $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{3}{4} \ldots$.

$\square$

$\mathbb{Q}$ is Countable

Proof.

- List the rationals in order by traversing the table as before, but skip over the unreduced rationals.
- The list begins $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{3}{4} \ldots$.
- An equivalent description of the list is
  - Arrange the rationals m/n in order of increasing sums $m + n$, leaving out the unreduced fractions.
  - Within groups, arrange the rationals m/n in order of increasing numerator.

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$\mathbb{Q}$ is Countable

Proof.

- List the rationals in order by traversing the table as before, but skip over the unreduced rationals.
- The list begins $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{3}{4} \ldots$.
- An equivalent description of the list is
  - Arrange the rationals m/n in order of increasing sums $m + n$, leaving out the unreduced fractions.
  - Within groups, arrange the rationals m/n in order of increasing numerator.
- Thus, $\mathbb{Q}$ is countable.

□

Outline

$\mathbb{R}$ is Uncountable

Theorem

$\mathbb{R}$ *is uncountable.*

Proof.

- We will use proof by contradiction because the definition of uncountable says that there *does not exist* a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{R}$.

□

Proof.

- We will use proof by contradiction because the definition of uncountable says that there *does not exist* a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{R}$.
- Suppose that $\mathbb{R}$ is countable.

$\square$

Proof.

- We will use proof by contradiction because the definition of uncountable says that there *does not exist* a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{R}$.
- Suppose that $\mathbb{R}$ is countable.
- Then there exists a one-to-one correspondence $f\colon \mathbb{N} \to \mathbb{R}$.

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Proof.

- We will use proof by contradiction because the definition of uncountable says that there *does not exist* a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{R}$.
- Suppose that $\mathbb{R}$ is countable.
- Then there exists a one-to-one correspondence $f : \mathbb{N} \to \mathbb{R}$.
- This correspondence allows us to list the real numbers $x_1, x_2, x_3, \ldots$

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Proof.

- We will use proof by contradiction because the definition of uncountable says that there *does not exist* a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{R}$.
- Suppose that $\mathbb{R}$ is countable.
- Then there exists a one-to-one correspondence $f : \mathbb{N} \to \mathbb{R}$.
- This correspondence allows us to list the real numbers $x_1, x_2, x_3, \ldots$
- Write each x_i in its decimal representation:

$$x_i = N_i.a_{i,1}a_{i,2}a_{i,3}\ldots a_{i,n}\ldots$$

where N_i is the integer part and each $a_{i,j}$ is one of the decimal digits of x_i.

□

Proof.

- We may list the decimal representations as follows:

$$
\begin{aligned}
x_1 &= N_1.a_{1,1}a_{1,2}a_{1,3}\ldots a_{1,n}\ldots\\
x_2 &= N_2.a_{2,1}a_{2,2}a_{2,3}\ldots a_{2,n}\ldots\\
x_3 &= N_3.a_{3,1}a_{3,2}a_{3,3}\ldots a_{3,n}\ldots\\
&\vdots\\
x_n &= N_n.a_{n,1}a_{n,2}a_{n,3}\ldots a_{n,n}\ldots\\
&\vdots
\end{aligned}
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□

Proof.

- Create a real number $x = 0.b_1 b_2 b_3 \ldots b_n \ldots$ as follows:

$$b_i = \begin{cases} 1 & \text{if } a_{i,i} = 0 \\ 0 & \text{if } a_{i,i} \neq 0. \end{cases}$$

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- Then x is a real number, but it is nowhere in the list because it differs from every x_i in position i.

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- Then x is a real number, but it is nowhere in the list because it differs from every x_i in position i.
- This is a contradiction.
- Therefore, $\mathbb{R}$ is uncountable.

□

Another Uncountable Set

Theorem

The set of all infinite binary sequences is uncountable.

Another Uncountable Set

Proof.

- Let B be the set of all infinite binary sequences.

□

Another Uncountable Set

Proof.

- Let B be the set of all infinite binary sequences.
- Suppose that B is countable.

□

Another Uncountable Set

Proof.

- Let B be the set of all infinite binary sequences.
- Suppose that B is countable.
- Then there is a one-to-one correspondence $f : \mathbb{N} \to B$.

$\square$

Proof.

- Let B be the set of all infinite binary sequences.
- Suppose that B is countable.
- Then there is a one-to-one correspondence $f : \mathbb{N} \to B$.
- List, as before, the images of $f(1), f(2), f(3), \ldots$ in a column.

□

Another Uncountable Set

Proof.

- We get something like the following (for example):

$f(i)$	b_1	b_2	b_3	b_4	$\cdots$	b_n	$\cdots$
$f(1)$	1	0	0	1	$\cdots$	0	$\cdots$
$f(2)$	0	0	1	1	$\cdots$	1	$\cdots$
$f(3)$	0	1	1	0	$\cdots$	1	$\cdots$
$f(4)$	1	1	0	1	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	
$f(n)$	1	1	1	0	$\cdots$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

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Another Uncountable Set

Proof.

- We get something like the following (for example):

$f(i)$	b_1	b_2	b_3	b_4	$\cdots$	b_n	$\cdots$
$f(1)$	**1**	0	0	1	$\cdots$	0	$\cdots$
$f(2)$	0	**0**	1	1	$\cdots$	1	$\cdots$
$f(3)$	0	1	**1**	0	$\cdots$	1	$\cdots$
$f(4)$	1	1	0	**1**	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	
$f(n)$	1	1	1	0	$\cdots$	**0**	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

□

Another Uncountable Set

Proof.

- Create a binary sequence b by reversing the binary digits on the diagonal.

□

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- Create a binary sequence b by reversing the binary digits on the diagonal.
- In the example, the diagonal is $1011\ldots0\ldots$.

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- Create a binary sequence b by reversing the binary digits on the diagonal.
- In the example, the diagonal is $1011\ldots0\ldots$.
- So, we let $b = 0100\ldots1\ldots$.

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Proof.

- Create a binary sequence b by reversing the binary digits on the diagonal.
- In the example, the diagonal is $1011\ldots0\ldots$.
- So, we let $b = 0100\ldots1\ldots$.
- Clearly, $b \neq f(n)$ for any $n \in \mathbb{N}$.

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- In the example, the diagonal is $1011\ldots0\ldots$.
- So, we let $b = 0100\ldots1\ldots$.
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- Therefore, f is not onto, which is a contradiction.

$\square$

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- In the example, the diagonal is $1011\ldots0\ldots$.
- So, we let $b = 0100\ldots1\ldots$.
- Clearly, $b \neq f(n)$ for any $n \in \mathbb{N}$.
- Therefore, f is not onto, which is a contradiction.
- Thus, B is uncountable.

□

Yet Another Uncountable Set

Example

- Let $\mathcal{F}$ be the set of all functions $f : \mathbb{N} \to \mathbb{N}$.

Yet Another Uncountable Set

Example

- Let $\mathcal{F}$ be the set of all functions $f : \mathbb{N} \to \mathbb{N}$.
- Then $\mathcal{F}$ is uncountable.

Outline

Lesser and Greater Cardinalities

Definition (Lesser and Greater Cardinalities)

A set A has a lesser cardinality than a set B, denoted $|A| < |B|$, if there does *not* exist an onto function from A to B. The set A has a greater cardinality than B, denoted $|A| > |B|$, if there does not exists a one-to-one function from A to B.

Lesser and Greater Cardinalities

Theorem

If a set A has a lesser cardinality than a set B, then the set B has a greater cardinality than the set A.

Lesser and Greater Cardinalities

Proof.

- We will use proof by contradiction.

□

Lesser and Greater Cardinalities

Proof.

- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.

□

Lesser and Greater Cardinalities

Proof.

- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.
- Then there is not an onto function A to B, but there is a one-to-one function f from B to A.

□

Proof.

- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.
- Then there is not an onto function A to B, but there is a one-to-one function f from B to A.
- Its image is $f[B] \subseteq A$.

□

Lesser and Greater Cardinalities

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- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.
- Then there is not an onto function A to B, but there is a one-to-one function f from B to A.
- Its image is $f[B] \subseteq A$.
- Clearly, $f\colon B \to f[B]$ is a one-to-one correspondence.

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- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.
- Then there is not an onto function A to B, but there is a one-to-one function f from B to A.
- Its image is $f[B] \subseteq A$.
- Clearly, $f\colon B \to f[B]$ is a one-to-one correspondence.
- Therefore, $f^{-1}\colon f[B] \to B$ is also a one-to-one correspondence, and thus onto B.

□

Lesser and Greater Cardinalities

Proof.

- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.
- Then there is not an onto function A to B, but there is a one-to-one function f from B to A.
- Its image is $f[B] \subseteq A$.
- Clearly, $f : B \to f[B]$ is a one-to-one correspondence.
- Therefore, $f^{-1} : f[B] \to B$ is also a one-to-one correspondence, and thus onto B.
- Extend f^{-1} from $f[B]$ to all of A by defining it arbitrarily on $A \setminus f[B]$.

□

Lesser and Greater Cardinalities

Proof.

- We will use proof by contradiction.
- Suppose that $|A| < |B|$, but that $|B| \not> |A|$.
- Then there is not an onto function A to B, but there is a one-to-one function f from B to A.
- Its image is $f[B] \subseteq A$.
- Clearly, $f : B \to f[B]$ is a one-to-one correspondence.
- Therefore, $f^{-1} : f[B] \to B$ is also a one-to-one correspondence, and thus onto B.
- Extend f^{-1} from $f[B]$ to all of A by defining it arbitrarily on $A \setminus f[B]$.
- The result is an onto function from A to B, a contradiction.

□

Power Sets

Theorem

For any set A, $|A| < |\mathcal{P}(A)|$.

Power Sets

Proof.

- We will use proof by contradiction because the definition of $<$ says that there *does not exist* an onto function from A to $\mathcal{P}(A)$.

□

Power Sets

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- We will use proof by contradiction because the definition of $<$ says that there *does not exist* an onto function from A to $\mathcal{P}(A)$.
- Suppose that there exists an onto function $f\colon A \to \mathcal{P}(A)$.

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Proof.

- We will use proof by contradiction because the definition of $<$ says that there *does not exist* an onto function from A to $\mathcal{P}(A)$.
- Suppose that there exists an onto function $f\colon A \to \mathcal{P}(A)$.
- Then for every subset $S \subseteq A$, there exists an element $a \in A$ such that $S = f(a)$.

□

Proof.

- We will use proof by contradiction because the definition of $<$ says that there *does not exist* an onto function from A to $\mathcal{P}(A)$.
- Suppose that there exists an onto function $f\colon A \to \mathcal{P}(A)$.
- Then for every subset $S \subseteq A$, there exists an element $a \in A$ such that $S = f(a)$.
- Create a set $S' \subseteq A$ as follows: for each $a \in A$,

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- Then for every subset $S \subseteq A$, there exists an element $a \in A$ such that $S = f(a)$.
- Create a set $S' \subseteq A$ as follows: for each $a \in A$,
  - if $a \in f(a)$, then $a \notin S'$, and

$\square$

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- Then for every subset $S \subseteq A$, there exists an element $a \in A$ such that $S = f(a)$.
- Create a set $S' \subseteq A$ as follows: for each $a \in A$,
  - if $a \in f(a)$, then $a \notin S'$, and
  - if $a \notin f(a)$, then $a \in S'$.
- Can $S' = f(a)$ for any $a \in A$?

□

Power Sets

Proof.

- We will use proof by contradiction because the definition of $<$ says that there *does not exist* an onto function from A to $\mathcal{P}(A)$.
- Suppose that there exists an onto function $f\colon A \to \mathcal{P}(A)$.
- Then for every subset $S \subseteq A$, there exists an element $a \in A$ such that $S = f(a)$.
- Create a set $S' \subseteq A$ as follows: for each $a \in A$,
    - if $a \in f(a)$, then $a \notin S'$, and
    - if $a \notin f(a)$, then $a \in S'$.
- Can $S' = f(a)$ for any $a \in A$?
- If not, then we have a contradiction and f is not onto and, therefore, $|A| < |\mathcal{P}(A)|$.

□

Levels of Infinity

- Given that $|A| < |\mathcal{P}(A)|$ for all sets A, we begin with $A = \mathbb{N}$.
  - $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$
  - $|\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$
  - $|\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$
  - And so on...

Levels of Infinity

Definition

Let

$$A_0 = \mathbb{N},$$
$$A_n = \mathcal{P}(A_{n-1}) \text{ for } n \geq 1.$$

Then define $\aleph_n = |A_n|$.

- It follows that

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_n < \cdots$$

Assignment

Assignment

- Presentations for Apr 20 and 22: 4.79, 4.81, 4.83, 4.86, 4.93.
- Write up Problem 4.80 and either Theorem 4.90 or Theorem 4.91 to be turned in on Wednesday, April 20.