Elementary Logic and Proof

Lecture 5

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Outline

Outline

Statements

Definition (Statement)

A proposition is a sentence that is either true or false.

- See the handout for examples.

Outline

Logical Operators

Definition (Conjunction)

The conjunction operator, denoted $\wedge$, joins two statements together into one statement. The conjunction is true if both statements are true. Otherwise, the conjunction is false.

Definition (Disjunction)

The disjunction operator, denoted $\vee$, joins two statements together into one statement. The disjunction is false if both statements are false. Otherwise, the disjunction is true.

- The English equivalent of conjunction is “and.”
- The English equivalent of disjunction is “or.”

Logical Operators

Definition (Negation)

The negation operator, denoted $\neg$, acts on a single statement. The negation is true if the statement is false. The negation is false if the statement is true.

- The English equivalent of negation is "not."

Conditional Statements

Definition (Conditional Statements)

The conditional operator, denoted $\Rightarrow$, joins two statements together into one statement. The conditional is false if the first statement is true and the second statement is false. Otherwise, the conditional statement is true.

- The English equivalent of the conditional is "if ... then ... ."

Outline

The Logical Symbols

- The LaTeX logical operators:
  - Conjuntion – `\wedge`
  - Disjuntion – `\vee`
  - Negation – `\neg`
  - Conditional – `\Rightarrow` (Note the capital `R`.)

Outline

Truth Tables

Definition (Truth Tables)

A truth table for a logical expression is a table that contains every possible combination of values of the variables together with the corresponding value of the expression.

- If there are n variables, then there are 2^n possible combinations of values.

Example

Example (Truth Tables)

Write a truth table for the expression $(A \wedge B) \vee C$.

A	B	C	$A \wedge B$	$(A \wedge B) \vee C$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Truth Tables

- Is
$$(A \wedge B) \vee C$$
logically equivalent to
$$A \wedge (B \vee C).$$
- How can we tell?

Outline

- The `tabular` environment is use to create tables.
- It is delimited by

```
\begin{tabular}{parameters}
        ⋮
\end{tabular}
```

The `tabular` Environment

- The parameters tell whether to left justify, center, or right justify the columns and where to draw vertical lines to separate the columns.
  - `l` – left justify (for text)
  - `c` – center (for whatever)
  - `r` – right justify (for numbers)
- Intersperse those letters with the vertical stroke | to draw a vertical line in the table.
- `\\` will start a new line.
- `\hline` will draw a horizontal line between rows.
- `&` advances to the next column.

Example

Example (Truth Table)

```
\begin{tabular}{|c|c||c|}
\hline
$A$ & $B$ & $A\wedge B$ \\
\hline\hline
T   & T   & T \\
T   & F   & F \\
F   & T   & F \\
F   & F   & F \\
\hline
\end{tabular}
```

Example

Example (Truth Table)

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Outline

Direct Proof

- A direct proof of the proposition $P \Rightarrow Q$
  - Begins by assuming that P is true.
  - Proceeds to draw logical deductions from that assumption.
  - Concludes by deducing that Q is true.

Example

Theorem

If n is a multiple of 20, then n is a multiple of 4 and n is a multiple of 5.

Direct Proof

Proof.

- Let n be a multiple of 20.

□

Direct Proof

Proof.

- Let n be a multiple of 20.
- Then $n = 20k$ for some integer k.

□

Proof.

- Let n be a multiple of 20.
- Then $n = 20k$ for some integer k.
- Then $n = 4(5k)$ and $n = 5(4k)$.

□

Direct Proof

Proof.

- Let n be a multiple of 20.
- Then $n = 20k$ for some integer k.
- Then $n = 4(5k)$ and $n = 5(4k)$.
- Therefore, n is a multiple of 4 and n is a multiple of 5.

□

Outline

Example

Theorem

If n is not a multiple of 20, then n is not a multiple of 4 or n is not a multiple of 5.

Proof by Contraposition

- The contrapositive of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.
- These two statements are logically equivalent.
- Therefore, if we can prove one of them, we have proven both of them.
- A proof by contraposition is a direct proof of the contrapositive.

Example

Example (Contrapositive)

- Let A be the statement "n is a multiple of 4."
- Let B be the statement "n is a multiple of 5."
- Let C be the statement "n is a multiple of 20."
- The theorem is of the form

$$\neg C \Rightarrow (\neg A \vee \neg B).$$

Example

Example (Contrapositive)

- The contrapositive is

$$\neg(\neg A \vee \neg B) \Rightarrow \neg(\neg C),$$

which is the same as

$$(A \wedge B) \Rightarrow C.$$

Example

Theorem

If n is a multiple of 4 and n is a multiple of 5, then n is a multiple of 20.

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.

□

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.

□

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.
- Notice that $1 = 16 - 15 = 4 \cdot 4 - 3 \cdot 5$.

□

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.
- Notice that $1 = 16 - 15 = 4 \cdot 4 - 3 \cdot 5$.
- Then

$$n = 16n - 15n$$

□

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.
- Notice that $1 = 16 - 15 = 4 \cdot 4 - 3 \cdot 5$.
- Then

$$\begin{aligned} n &= 16n - 15n \\ &= 4(4n) - 3(5n) \end{aligned}$$

□

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.
- Notice that $1 = 16 - 15 = 4 \cdot 4 - 3 \cdot 5$.
- Then

$$\begin{aligned} n &= 16n - 15n \\ &= 4(4n) - 3(5n) \\ &= 4(20m) - 3(20k) \end{aligned}$$

□

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.
- Notice that $1 = 16 - 15 = 4 \cdot 4 - 3 \cdot 5$.
- Then

$$\begin{aligned} n &= 16n - 15n \\ &= 4(4n) - 3(5n) \\ &= 4(20m) - 3(20k) \\ &= 20(4m - 3k). \end{aligned}$$

□

Proof by Contraposition

Proof.

- Let n be a multiple of 4 and also a multiple of 5.
- Then $n = 4k$ for some integer k and $n = 5m$ for some integer m.
- Notice that $1 = 16 - 15 = 4 \cdot 4 - 3 \cdot 5$.
- Then

$$\begin{aligned} n &= 16n - 15n \\ &= 4(4n) - 3(5n) \\ &= 4(20m) - 3(20k) \\ &= 20(4m - 3k). \end{aligned}$$

- Therefore, n is a multiple of 20.

□

Example

Example (Contrapositive)

- The contrapositive is

$$\neg(\neg A \vee \neg B) \Rightarrow \neg(\neg C),$$

which is the same as

$$(A \wedge B) \Rightarrow C.$$

Example

Corollary

If n is a multiple of 4 and n is ***not*** *a multiple of 20, then n is* ***not*** *a multiple of 5.*

Proof by Partial Contraposition

Proof.

- The partial contrapositive of this theorem is the statement: "If n is a multiple of 4 and n **is** a multiple of 5, then n **is** a multiple of 20.

□

Proof by Partial Contraposition

Proof.

- The partial contrapositive of this theorem is the statement: “If n is a multiple of 4 and n **is** a multiple of 5, then n **is** a multiple of 20.
- Thus, it follows immediately from the previous theorem.

□