Proof by Contradiction

Lecture 6

Robb T. Koether

Hampden-Sydney College

Mon, Feb 13, 2017

Outline

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1. Proof by Contradiction

Proof by Contradiction

- Every statement must be either true or false.
- It is impossible to correctly deduce a false statement from a true statement.
- A false statement can be deduced only from a false statement.
- This is the basis of proof by contradiction: If the supposition $\neg P$ leads to a false statement, then $\neg P$ must be false, and therefore P must be true.

Proof by Contradiction

- To prove the statement $P \Rightarrow Q$ by contradiction, we suppose that its negation is true and then deduce a false statement from that negation.
- The negation of $P \Rightarrow Q$ is $P \wedge \neg Q$.
- To prove by contradiction that $P \Rightarrow Q$, we suppose that P is true and Q is false and then reach a contradiction (a false statement).

Example

Theorem

$\sqrt{2}$ *is irrational.*

Example

Theorem

Let x be a real number. If $x^2 = 2$, then $x \neq \frac{a}{b}$ for any choice of integers a and b.

- Let P be the statement "$x^2 = 2$."
- Let Q be the statement "$x \neq \frac{a}{b}$ for any choice of integers a and b."
- Then $\neg Q$ is the statement "$x = \frac{a}{b}$ for some choice of integers a and b."

Proof by Contradiction

Proof.

- We will use proof by contradiction.

□

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- We will use proof by contradiction.
- Let be x a real number and suppose that $x^2 = 2$ and $x = \frac{a}{b}$ for some choice of integers a and b.

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Proof by Contradiction

Proof.

- We will use proof by contradiction.
- Let be x a real number and suppose that $x^2 = 2$ and $x = \frac{a}{b}$ for some choice of integers a and b.
- Assume that a and b have no factor in common.

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- Then

$$bx = a,$$

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- Assume that a and b have no factor in common.
- Then

$$
\begin{aligned}
bx &= a, \\
b^2x^2 &= a^2,
\end{aligned}
$$

□

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- Let be x a real number and suppose that $x^2 = 2$ and $x = \frac{a}{b}$ for some choice of integers a and b.
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$$\begin{aligned} bx &= a, \\ b^2x^2 &= a^2, \\ 2b^2 &= a^2. \end{aligned}$$

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- We will use proof by contradiction.
- Let be x a real number and suppose that $x^2 = 2$ and $x = \frac{a}{b}$ for some choice of integers a and b.
- Assume that a and b have no factor in common.
- Then

$$\begin{aligned} bx &= a, \\ b^2x^2 &= a^2, \\ 2b^2 &= a^2. \end{aligned}$$

- Then a^2 is even and it follows that a is even. (Why?)

□

Proof by Contradiction

Proof.

- Let $a = 2k$ for some integer k.

□

Proof by Contradiction

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- Let $a = 2k$ for some integer k.
- Then

$\square$

Proof by Contradiction

Proof.

- Let $a = 2k$ for some integer k.
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$$2b^2 = (2k)^2,$$

□

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- Let $a = 2k$ for some integer k.
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2b^2 &= (2k)^2, \\
2b^2 &= 4k^2, \\
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- By the same reasoning as before, b must be even.

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$$2b^2 = (2k)^2,$$
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- By the same reasoning as before, b must be even.
- But a and b have no factor in common.

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- But a and b have no factor in common.
- That is a contradiction.

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- By the same reasoning as before, b must be even.
- But a and b have no factor in common.
- That is a contradiction.
- Therefore, $\sqrt{2}$ is irrational.

□

Theorem

Let a, b, c, and d be integers. If $a + b + c + d$ is odd, then a is odd or b is odd or c is odd or d is odd.

Contradiction vs. Contraposition

Contradiction vs. Contraposition

- Suppose we want to prove by contradiction that $A \Rightarrow B$.

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- From that we derive the contradiction $A \wedge \neg A$ and conclude that $A \Rightarrow B$.

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- Suppose we want to prove by contradiction that $A \Rightarrow B$.
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- How does this differ from proof by contraposition?

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- Suppose we want to prove by contradiction that $A \Rightarrow B$.
- We begin by supposing that $A \wedge \neg B$.
- From that we derive the contradiction $A \wedge \neg A$ and conclude that $A \Rightarrow B$.
- How does this differ from proof by contraposition?
- Which method is to be preferred?

Example

Theorem

Let a and n be integers. If $a > 1$ and $a \mid n$, then $a \nmid n + 1$.

Example

Theorem

There are infinitely many primes.