Quantifiers

Lecture 7

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Outline

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Free Variables

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- For example, in the sentence "x is divisible by 4", x is a free variable.
- If a sentence contains a free variable, then the sentence cannot be either true or false.

Predicates

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- For example, the sentence “x is divisible by 4” is a predicate.
- The sentence “If $x = 8$, then x is divisible by 4” is a proposition, not a predicate.
- The sentence “If $x = 7$, then x is divisible by 4” is also a proposition.

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- For example, in the sentence "There is an integer x that is divisible by 4", x is a bound variable.
- If all the variables in a sentence are bound, then the sentence is either true or false.

Outline

The Quantifiers

- We use two quantifiers to bind variables.
- The universal quantifier uses the phrase

  "for all ... in the set ..."
- The existential quantifier uses the phrase

  "there exists ... in the set ... such that ..."
- We use the symbols $\forall$ and $\exists$ to represent the universal and existential quantifiers, respectively.

Outline

A Little More $\LaTeX$

- The quantifiers (in math mode)
  - For every – `\forall` will produce $\forall$.
  - There exists – `\exists` will produce $\exists$.
- Set membership (in math mode)
  - Is an element of – `\in` will produce $\in$.
  - Is not an element of – `\notin` will produce $\notin$.
- Standard sets of numbers (in math mode)
  - Natural numbers – `\mathbb{N}` will produce $\mathbb{N}$.
  - Integers – `\mathbb{Z}` will produce $\mathbb{Z}$.
  - Rational numbers – `\mathbb{Q}` will produce $\mathbb{Q}$.
  - Real numbers – `\mathbb{R}` will produce $\mathbb{R}$.
  - Complex numbers – `\mathbb{C}` will produce $\mathbb{C}$.

Example

Example (A Little More LaTeX)

- The LaTeX expression

```
(\forall n\in\mathbb{Z})(\exists
    m\in\mathbb{Z})(n+m=0)
```

produces

$$(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})(n + m = 0)$$

Outline

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Example (Quantifiers)

- Write each sentence using the qualifier symbols and then identify the sentence as (1) a true statement, (2) a false statement, or (3) neither.
  1. For all real numbers x, x^2 is positive.
     - Equivalently, For all x in the set $\mathbb{R}$, x^2 is positive.

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  5. For every real number x, there exists a real number y such that $x + y = 1$.
  6. There exists real number x such that for every a real number y, $x + y = 1$.
  7. There exists an integer a such that for every integer b there exists an integer c such that $a + bc \neq 0$.

The Quantifiers

Example (Quantifiers)

- Consider "There exists an integer a such that for every integer b there exists an integer c such that $a + bc \neq 0$."

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- Consider "There exists an integer a such that for every integer b there exists an integer c such that $a + bc \neq 0$."
- $(\exists a \in \mathbb{Z})(\forall b \in \mathbb{Z})(\exists c \in \mathbb{Z})(a + bc \neq 0)$.

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  - You choose a, hoping to make $a + bc \neq 0$.

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  - You choose a, hoping to make $a + bc \neq 0$.
  - Your worst enemy chooses b, hoping to force $a + bc = 0$.

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- $(\exists a \in \mathbb{Z})(\forall b \in \mathbb{Z})(\exists c \in \mathbb{Z})(a + bc \neq 0)$.
  - You choose a, hoping to make $a + bc \neq 0$.
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  - You choose c, hoping to make $a + bc \neq 0$.

Example (Quantifiers)

- Consider "There exists an integer *a* such that for every integer *b* there exists an integer *c* such that $a + bc \neq 0$."
- $(\exists a \in \mathbb{Z})(\forall b \in \mathbb{Z})(\exists c \in \mathbb{Z})(a + bc \neq 0)$.
  - You choose *a*, hoping to make $a + bc \neq 0$.
  - Your worst enemy chooses *b*, hoping to force $a + bc = 0$.
  - You choose *c*, hoping to make $a + bc \neq 0$.
- Can you do it?
- Can your worst enemy prevent you from doing it?

The Order of Quantifiers

- Note that the order of the quantifiers $\forall$ and $\exists$ cannot be changed without changing the meaning of the statement.
- However, the order of consecutive occurrences of $\forall$ or consecutive occurrences of $\exists$ can be reversed without changing the meaning of the statement.

The Order of Quantifiers

Example

- The statements

$$(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})(a + b \in \mathbb{Z})$$

  and

$$(\forall b \in \mathbb{Z})(\forall a \in \mathbb{Z})(a + b \in \mathbb{Z})$$

  are logically equivalent.

Example

- The statements

$$(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})(ab = 6)$$

and

$$(\exists b \in \mathbb{Z})(\exists a \in \mathbb{Z})(ab = 6)$$

are logically equivalent.

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- If a proposition is not true for every x in a set, then it must be false for at least one x in that set.

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- Therefore,

$$\neg\left(\forall x \in S\right)\left(P(x)\right)$$

is equivalent to

$$\left(\exists x \in S\right)\left(\neg P(x)\right).$$

- Similarly, if a proposition is not true for even one x in a set, then it must be false for all x in that set.

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- Therefore,

$$\neg (\exists x \in S)(P(x))$$

is equivalent to

$$(\forall x \in S)(\neg P(x)).$$

- Write the negations of the following propositions and then determine which is true, the proposition or its negation.

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    1. $(\forall a \in \mathbb{Z})\,(\forall b \in \mathbb{Z})\,((ab \text{ is even}) \Rightarrow ((a \text{ is even}) \vee (b \text{ is even})))$.

- Write the negations of the following propositions and then determine which is true, the proposition or its negation.
  1. $(\forall a \in \mathbb{Z})\,(\forall b \in \mathbb{Z})\,((ab \text{ is even}) \Rightarrow ((a \text{ is even}) \vee (b \text{ is even})))$.
  2. $(\forall a \in \mathbb{Z})\,(\forall b \in \mathbb{Z})\,((4 \mid ab) \Rightarrow (4 \mid a) \vee (4 \mid b)))$.

- Write the negations of the following propositions and then determine which is true, the proposition or its negation.
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  2. $(\forall a \in \mathbb{Z})\,(\forall b \in \mathbb{Z})\,((4 \mid ab) \Rightarrow (4 \mid a) \vee (4 \mid b)))$.
  3. $(\exists a \in \mathbb{Z})\,(\exists b \in \mathbb{Z})\,\left(a^2 + b^2 = 99\right)$.

- Write the negations of the following propositions and then determine which is true, the proposition or its negation.
    1. $(\forall a \in \mathbb{Z})\,(\forall b \in \mathbb{Z})\,((ab \text{ is even}) \Rightarrow ((a \text{ is even}) \vee (b \text{ is even})))$.
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    3. $(\exists a \in \mathbb{Z})\,(\exists b \in \mathbb{Z})\,\left(a^2 + b^2 = 99\right)$.
    4. $(\exists a \in \mathbb{Z})\,(\exists b \in \mathbb{Z})\,\left(a^2 - b^2 = 99\right)$.

- Write the negations of the following propositions and then determine which is true, the proposition or its negation.
  1. $(\forall a \in \mathbb{Z})\,(\forall b \in \mathbb{Z})\,((ab \text{ is even}) \Rightarrow ((a \text{ is even}) \vee (b \text{ is even})))$.
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  3. $(\exists a \in \mathbb{Z})\,(\exists b \in \mathbb{Z})\,\left(a^2 + b^2 = 99\right)$.
  4. $(\exists a \in \mathbb{Z})\,(\exists b \in \mathbb{Z})\,\left(a^2 - b^2 = 99\right)$.
  5. $(\forall a \in \mathbb{Z})\,(\exists b \in \mathbb{Z})\,(\exists c \in \mathbb{Z})\,(a = 2b + 3c)$.

Outline

Proving a Universal Statement

- To prove directly a statement of the form "$(\forall x \in S)\,(P(x))$," begin with "Let $x \in S$."
- Do not assume anything about x except that it is a member of the set S.
- Then go on to argue that $P(x)$ is true.

Proving an Existential Statement

- To prove directly a statement of the form "$(\exists x \in S)\,(P(x))$," you can proceed in one of two ways.
  - Produce a specific value of x for which $P(x)$ is true.
  - Argue indirectly that such an x must exist.