Sets

Lecture 8

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Outline

Outline

Set Builder Notation

- Set builder notation defines a set by using the following form.

$$S = \{x \in A \mid x \text{ satisfies a specific condition}\}.$$

- Examples
  - The set of all positive real numbers.
  - The set of all even integers.
  - The set of all real solutions to the equation $x^5 + x - 1 = 0$.
  - The set of all integer solutions to the equation $x^5 + x - 1 = 0$.
  - The set of all primes.

Subsets

Definition (Subset)

A set A is a subset of a set B, denoted $A \subseteq B$, if every element of A is also an element of B.

Definition (Proper Subset)

A set A is a proper subset of a set B, denoted $A \subset B$, if every element of A is also an element of B, but $A \neq B$.

Subsets

- The statement $A \subseteq B$ can be expressed as a conditional:

$$x \in A \Rightarrow x \in B.$$

- The difference between “subset” and “proper subset” is analogous to the difference between $\leq$ and $<$.

The Empty Set

Definition (The Empty Set)

The empty set, or null set, denoted $\varnothing$ or $\{\}$, is the set that contains no elements.

- The empty set is a subset of every set.

Set Operations

Let U be the universe of discourse.

Definition (Intersection)

The intersection of sets A and B is the set

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}.$$

Definition (Union)

The union of sets A and B is the set

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}.$$

Set Operations

Let U be the universe of discourse.

Definition (Complement)

The complement of a set A is the set

$$A^c = \{x \in U \mid x \notin A\}.$$

Definition (Set Difference)

The difference of sets A and B is the set

$$A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\}.$$

Outline

Some LaTeX

Math	LaTeX	Symbol
Set builder	`\{`	$\{$
Set builder	`\}`	$\}$
Set builder	`\mid`	$\mid$
Subset	`\subseteq`	$\subseteq$
Proper subset	`\subset`	$\subset$
Empty set (ugly)	`\emptyset`	$\emptyset$
Empty set (pretty)	`\varnothing`	$\varnothing$
Union	`\cup`	$\cup$
Intersection	`\cap`	$\cap$
Complement	`^c`	^{c}
Set difference	`\setminus`	$\setminus$

Some LaTeX

- The code

```
A\setminus B\subseteq(A\cap B)^c
```

  will produce

$$A \setminus B \subseteq (A \cap B)^c.$$

Outline

Proofs Involving Sets

Theorem

For any sets A, B, and C, if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

Proof.

- Let A, B, and C be sets.

□

Proof.

- Let A, B, and C be sets.
- Suppose that $A \subseteq B$.

□

Proof.

- Let A, B, and C be sets.
- Suppose that $A \subseteq B$.
- Let $x \in A \cap C$.

□

Proof.

- Let A, B, and C be sets.
- Suppose that $A \subseteq B$.
- Let $x \in A \cap C$.
- Then $x \in A$ and $x \in C$.

$\square$

Proof.

- Let A, B, and C be sets.
- Suppose that $A \subseteq B$.
- Let $x \in A \cap C$.
- Then $x \in A$ and $x \in C$.
- Because $A \subseteq B$, it follows that $x \in B$.

$\square$

Proof.

- Let A, B, and C be sets.
- Suppose that $A \subseteq B$.
- Let $x \in A \cap C$.
- Then $x \in A$ and $x \in C$.
- Because $A \subseteq B$, it follows that $x \in B$.
- Therefore, $x \in B \cap C$.

$\square$

Proof.

- Let A, B, and C be sets.
- Suppose that $A \subseteq B$.
- Let $x \in A \cap C$.
- Then $x \in A$ and $x \in C$.
- Because $A \subseteq B$, it follows that $x \in B$.
- Therefore, $x \in B \cap C$.
- Thus, $A \cap C \subseteq B \cap C$.

$\square$

Proofs Involving Sets

Theorem

For any sets A, B, and C,

$$(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C.$$

Proofs Involving Sets

Proof.

- Let A, B, and C be sets.

□

Proofs Involving Sets

Proof.

- Let A, B, and C be sets.
- Let $x \in (A \setminus C) \cup (B \setminus C)$.

□

Proof.

- Let A, B, and C be sets.
- Let $x \in (A \setminus C) \cup (B \setminus C)$.
- Then $x \in A \setminus C$ or $x \in B \setminus C$.

$\square$

Proofs Involving Sets

Proof.

- Let A, B, and C be sets.
- Let $x \in (A \setminus C) \cup (B \setminus C)$.
- Then $x \in A \setminus C$ or $x \in B \setminus C$.
- Case 1: Suppose $x \in A \setminus C$.

□

Proof.

- Let A, B, and C be sets.
- Let $x \in (A \setminus C) \cup (B \setminus C)$.
- Then $x \in A \setminus C$ or $x \in B \setminus C$.
- Case 1: Suppose $x \in A \setminus C$.
  - Then $x \in A$ and $x \notin C$.

□

Proof.

- Let A, B, and C be sets.
- Let $x \in (A \setminus C) \cup (B \setminus C)$.
- Then $x \in A \setminus C$ or $x \in B \setminus C$.
- Case 1: Suppose $x \in A \setminus C$.
  - Then $x \in A$ and $x \notin C$.
  - Because $x \in A$, then $x \in A \cup B$.

□

Proofs Involving Sets

Proof.

- Let A, B, and C be sets.
- Let $x \in (A \setminus C) \cup (B \setminus C)$.
- Then $x \in A \setminus C$ or $x \in B \setminus C$.
- Case 1: Suppose $x \in A \setminus C$.
  - Then $x \in A$ and $x \notin C$.
  - Because $x \in A$, then $x \in A \cup B$.
  - Therefore, $x \in (A \cup B) \setminus C$.

□

Proofs Involving Sets

Proof.

- Case 2: Suppose $x \in B \setminus C$.

□

Proofs Involving Sets

Proof.

- Case 2: Suppose $x \in B \setminus C$.
  - Then $x \in B$ and $x \notin C$.

□

Proofs Involving Sets

Proof.

- Case 2: Suppose $x \in B \setminus C$.
  - Then $x \in B$ and $x \notin C$.
  - Because $x \in B$, then $x \in A \cup B$.

□

Proofs Involving Sets

Proof.

- Case 2: Suppose $x \in B \setminus C$.
  - Then $x \in B$ and $x \notin C$.
  - Because $x \in B$, then $x \in A \cup B$.
  - Therefore, $x \in (A \cup B) \setminus C$.

□

Proofs Involving Sets

Proof.

- Case 2: Suppose $x \in B \setminus C$.
  - Then $x \in B$ and $x \notin C$.
  - Because $x \in B$, then $x \in A \cup B$.
  - Therefore, $x \in (A \cup B) \setminus C$.
- Therefore, $x \in (A \cup B) \setminus C$.

□

Proofs Involving Sets

Proof.

- Case 2: Suppose $x \in B \setminus C$.
  - Then $x \in B$ and $x \notin C$.
  - Because $x \in B$, then $x \in A \cup B$.
  - Therefore, $x \in (A \cup B) \setminus C$.
- Therefore, $x \in (A \cup B) \setminus C$.
- If follows that $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$.

□

Proofs Involving Sets

- Now prove the converse.

Proofs Involving Sets

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- That is, prove that

$$(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C).$$

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- That is, prove that

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- Why is that the "converse?"

Proofs Involving Sets

- Now prove the converse.
- That is, prove that

$$(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C).$$

- Why is that the "converse?"
- After proving the converse, it follows that

$$(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.$$

Outline

Indexing Notation

- We are familiar with the indexing notation when working with sequences $x_1, x_2, x_3, \ldots$.
- The sequence can be abbreviated as

$$\{x_i\}_{i=1}^{\infty},$$

or more simply as

$$\{x_i\}_{i\in\mathbb{N}}.$$

- We represent the infinite sum $x_1 + x_2 + x_3 + \cdots$ as

$$\sum_{i=1}^{\infty} x_i \text{ or } \sum_{i\in\mathbb{N}} x_i.$$

Indexing Sets

- When working with a collection of sets, it is often convenient to do the same thing.
- For example, rather than write

$$A, B, C, \ldots,$$

we write

$$A_1, A_2, A_3, \ldots.$$

- This can be abbreviated as

$$\{A_i\}_{i=1}^{\infty} \text{ or } \{A_i\}_{i \in \mathbb{N}}.$$

Arbitrary Unions and Intersections

- Rather than “add” sets, we take their union and their intersection.
- The union of a collection of sets

$$A_1 \cup A_2 \cup A_3 \cup \cdots$$

is denoted

$$\bigcup_{i=1}^{\infty} A_i \text{ or } \bigcup_{i \in \mathbb{N}} A_i.$$

Arbitrary Unions and Intersections

- The intersection of a collection of sets

$$A_1 \cap A_2 \cap A_3 \cap \cdots$$

is denoted

$$\bigcap_{i=1}^{\infty} A_i \text{ or } \bigcap_{i \in \mathbb{N}} A_i.$$

Arbitrary Unions and Intersections

Definition (Arbitrary Union)

$$\bigcup_{i\in\mathbb{N}} A_i = \{x \in U \mid x \in A_i \text{ for some } i \in \mathbb{N}\}$$
$$= \{x \in U \mid (\exists i \in \mathbb{N})\,(x \in A_i)\}\,.$$

Definition (Arbitrary Intersection)

$$\bigcap_{i\in\mathbb{N}} A_i = \{x \in U \mid x \in A_i \text{ for every } i \in \mathbb{N}\}$$
$$= \{x \in U \mid (\forall i \in \mathbb{N})\,(x \in A_i)\}\,.$$

Outline

Some More LaTeX

Math	LaTeX	Symbol
Sum	\sum_{i=1}^\infty A_i	$\sum_{i=1}^{\infty} A_i$
Sum	\sum_{i\in\mathbb{N}}A_i	$\sum_{i\in\mathbb{N}} A_i$
Union	\bigcup_{i\in\mathbb{N}}A_i	$\bigcup_{i\in\mathbb{N}} A_i$
Intersection	\bigcap_{i\in\mathbb{N}}A_i	$\bigcap_{i\in\mathbb{N}} A_i$

Example

Example (Exercise 2.56)

- Let $T_n = \left\{x \in \mathbb{R} \middle| -\frac{1}{n} < x < \frac{1}{n}\right\} = \left(-\frac{1}{n}, \frac{1}{n}\right)$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} T_n$.
  - Describe $\bigcap_{n=1}^{\infty} T_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $M_n = \{k \in \mathbb{N} | k \text{ is a multiple of } n\}$, where $n \in \mathbb{N}$.
  - Describe $\displaystyle\bigcup_{n=1}^{\infty} M_n$.
  - Describe $\displaystyle\bigcap_{n=1}^{\infty} M_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $M_n = \{k \in \mathbb{N} | k \text{ is a multiple of } n\}$, where $n \in \mathbb{N}$.
  - Describe $\displaystyle\bigcup_{n=1}^{\infty} M_n$.
  - Describe $\displaystyle\bigcap_{n=1}^{\infty} M_n$.
  - Describe $\displaystyle\bigcap_{n=1}^{10} M_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $S_n = \{k \in \mathbb{N} | n \leq k \leq 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} S_n$.
  - Describe $\bigcap_{n=1}^{\infty} S_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $S_n = \{k \in \mathbb{N} | n \leq k \leq 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} S_n$.
  - Describe $\bigcap_{n=1}^{\infty} S_n$.
- Prove that $\bigcup_{n=1}^{\infty} S_n = \mathbb{N}$.

Example

Example (Arbitrary Union and Intersection)

- Let $S_n = \{k \in \mathbb{N} | n \leq k \leq 2n\}$, where $n \in \mathbb{N}$.
    - Describe $\displaystyle\bigcup_{n=1}^{\infty} S_n$.
    - Describe $\displaystyle\bigcap_{n=1}^{\infty} S_n$.
- Prove that $\displaystyle\bigcup_{n=1}^{\infty} S_n = \mathbb{N}$.
- Prove that $\displaystyle\bigcap_{n=1}^{\infty} S_n = \varnothing$.

Example

Example (Arbitrary Union and Intersection)

- Let $A_n = \{k \in \mathbb{N} \,|\, n^2 \leq k < n^2 + 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\displaystyle\bigcup_{n=1}^{\infty} A_n$.
  - Describe $\displaystyle\bigcap_{n=1}^{\infty} A_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $A_n = \left\{k \in \mathbb{N} \middle| n^2 \leq k < n^2 + 2n\right\}$, where $n \in \mathbb{N}$.
  - Describe $\displaystyle\bigcup_{n=1}^{\infty} A_n$.
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Example (Arbitrary Union and Intersection)

- Let $A_n = \{k \in \mathbb{N} \mid n^2 \le k < n^2 + 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} A_n$.
  - Describe $\bigcap_{n=1}^{\infty} A_n$.
- Prove that $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.
- Prove that $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

Outline

Proof

Theorem

Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set. Then

$$\bigcup_{i \in \mathbb{N}} (A_i \setminus B) = \left(\bigcup_{i \in \mathbb{N}} A_i \right) \setminus B$$

and

$$\bigcap_{i \in \mathbb{N}} (A_i \setminus B) = \left(\bigcap_{i \in \mathbb{N}} A_i \right) \setminus B$$

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.

$\square$

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.

$\square$

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.

□

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.

□

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- That is, $x \in \bigcap_{i \in \mathbb{N}} A_i$ and $x \notin B$.

□

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- That is, $x \in \bigcap_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

$\square$

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- That is, $x \in \bigcap_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- It follows that $\bigcup_{i \in \mathbb{N}} (A_i \setminus B) \subseteq \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

□

Proof

Proof.

- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

□

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.

$\square$

Proof

Proof.

- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.

$\square$

Proof

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.

□

Proof

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- It follows that $\left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B \subseteq \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.

□

Proof

Proof.

- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- It follows that $\left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B \subseteq \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Thus, $\bigcup_{i \in \mathbb{N}} (A_i \setminus B) = \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

□

Outline

Questions

- If $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$, what is $\bigcup_{i \in \mathbb{N}} A_i$?

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- If $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$, what is $\bigcup_{i \in \mathbb{N}} A_i$?
- If $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$, what is $\bigcap_{i \in \mathbb{N}} A_i$?

Questions

- If $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$, what is $\bigcup_{i \in \mathbb{N}} A_i$?
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Questions

- If $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$, what is $\bigcup_{i \in \mathbb{N}} A_i$?
- If $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$, what is $\bigcap_{i \in \mathbb{N}} A_i$?
- If $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$, what is $\bigcup_{i \in \mathbb{N}} A_i$?
- If $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$, what is $\bigcap_{i \in \mathbb{N}} A_i$?

Questions

$\left(\bigcup_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcup_{j\in\mathbb{N}} B_j\right)$	$\bigcup_{i\in\mathbb{N}} \left(\bigcup_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$
$\left(\bigcup_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcap_{j\in\mathbb{N}} B_j\right)$	$\bigcup_{i\in\mathbb{N}} \left(\bigcap_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$
$\left(\bigcap_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcup_{j\in\mathbb{N}} B_j\right)$	$\bigcap_{i\in\mathbb{N}} \left(\bigcup_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$
$\left(\bigcap_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcap_{j\in\mathbb{N}} B_j\right)$	$\bigcap_{i\in\mathbb{N}} \left(\bigcap_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$

Which sets are equal?