The Topology of $\mathbb{R}$

Lecture 9

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Outline

Outline

Indexing Notation

- We are familiar with the indexing notation when working with sequences $x_1, x_2, x_3, \ldots$.
- The sequence can be abbreviated as

$$\{x_i\}_{i=1}^{\infty},$$

  or more simply as

$$\{x_i\}_{i\in\mathbb{N}}.$$

- We represent the infinite sum $x_1 + x_2 + x_3 + \cdots$ as

$$\sum_{i=1}^{\infty} x_i \text{ or } \sum_{i\in\mathbb{N}} x_i.$$

Indexing Sets

- When working with a collection of sets, it is often convenient to do the same thing.
- For example, rather than write

$$A, B, C, \ldots,$$

  we write

$$A_1, A_2, A_3, \ldots.$$

- This can be abbreviated as

$$\{A_i\}_{i=1}^{\infty} \text{ or } \{A_i\}_{i \in \mathbb{N}}.$$

Arbitrary Unions and Intersections

- Rather than “add” sets, we take their union and their intersection.
- The union of a collection of sets

$$A_1 \cup A_2 \cup A_3 \cup \cdots$$

is denoted

$$\bigcup_{i=1}^{\infty} A_i \text{ or } \bigcup_{i \in \mathbb{N}} A_i.$$

Arbitrary Unions and Intersections

- The intersection of a collection of sets

$$A_1 \cap A_2 \cap A_3 \cap \cdots$$

is denoted

$$\bigcap_{i=1}^{\infty} A_i \text{ or } \bigcap_{i \in \mathbb{N}} A_i.$$

Arbitrary Unions and Intersections

Definition (Arbitrary Union)

$$\bigcup_{i \in \mathbb{N}} A_i = \{x \in U \mid x \in A_i \text{ for some } i \in \mathbb{N}\}$$
$$= \{x \in U \mid (\exists i \in \mathbb{N})\,(x \in A_i)\}.$$

Definition (Arbitrary Intersection)

$$\bigcap_{i \in \mathbb{N}} A_i = \{x \in U \mid x \in A_i \text{ for every } i \in \mathbb{N}\}$$
$$= \{x \in U \mid (\forall i \in \mathbb{N})\,(x \in A_i)\}.$$

Outline

Some More LaTeX

Math	LaTeX	Symbol
Sum	`\sum_{i=1}^\infty A_i`	$\sum_{i=1}^\infty A_i$
Sum	`\sum_{i\in\mathbb{N}}A_i`	$\sum_{i\in\mathbb{N}}A_i$
Union	`\bigcup_{i\in\mathbb{N}}A_i`	$\bigcup_{i\in\mathbb{N}}A_i$
Intersection	`\bigcap_{i\in\mathbb{N}}A_i`	$\bigcap_{i\in\mathbb{N}}A_i$

Example

Example (Exercise 2.56)

- Let $T_n = \{x \in \mathbb{R} | -\frac{1}{n} < x < \frac{1}{n}\} = (-\frac{1}{n}, \frac{1}{n})$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} T_n$.
  - Describe $\bigcap_{n=1}^{\infty} T_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $M_n = \{k \in \mathbb{N} | k \text{ is a multiple of } n\}$, where $n \in \mathbb{N}$.
    - Describe $\bigcup_{n=1}^{\infty} M_n$.
    - Describe $\bigcap_{n=1}^{\infty} M_n$.

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Example (Arbitrary Union and Intersection)

- Let $M_n = \{k \in \mathbb{N} | k \text{ is a multiple of } n\}$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} M_n$.
  - Describe $\bigcap_{n=1}^{\infty} M_n$.
  - Describe $\bigcap_{n=1}^{10} M_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $S_n = \{k \in \mathbb{N} | n \leq k \leq 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} S_n$.
  - Describe $\bigcap_{n=1}^{\infty} S_n$.

Example

Example (Arbitrary Union and Intersection)

- Let $S_n = \{k \in \mathbb{N} | n \leq k \leq 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\bigcup_{n=1}^{\infty} S_n$.
  - Describe $\bigcap_{n=1}^{\infty} S_n$.
- Prove that $\bigcup_{n=1}^{\infty} S_n = \mathbb{N}$.

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  - Describe $\bigcap_{n=1}^{\infty} S_n$.
- Prove that $\bigcup_{n=1}^{\infty} S_n = \mathbb{N}$.
- Prove that $\bigcap_{n=1}^{\infty} S_n = \varnothing$.

Example

Example (Arbitrary Union and Intersection)

- Let $A_n = \{k \in \mathbb{N} \mid n^2 \leq k < n^2 + 2n\}$, where $n \in \mathbb{N}$.
  - Describe $\displaystyle\bigcup_{n=1}^{\infty} A_n$.
  - Describe $\displaystyle\bigcap_{n=1}^{\infty} A_n$.

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  - Describe $\bigcup_{n=1}^{\infty} A_n$.
  - Describe $\bigcap_{n=1}^{\infty} A_n$.
- Prove that $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.
- Prove that $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

Outline

Proof

Theorem

Let $\{A_i\}_{i \in \mathbb{N}}$ *be a collection of sets and let* B *be a set. Then*

$$\bigcup_{i \in \mathbb{N}} (A_i \setminus B) = \left(\bigcup_{i \in \mathbb{N}} A_i \right) \setminus B$$

and

$$\bigcap_{i \in \mathbb{N}} (A_i \setminus B) = \left(\bigcap_{i \in \mathbb{N}} A_i \right) \setminus B$$

Proof

Proof.

- Let $\{A_i\}_{i\in\mathbb{N}}$ be a collection of sets and let B be a set.

□

Proof

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- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.

□

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.

$\square$

Proof

Proof.

- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
- Let $x \in \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.

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- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
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- Then $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- That is, $x \in \bigcap_{i \in \mathbb{N}} A_i$ and $x \notin B$.

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- Let $\{A_i\}_{i\in\mathbb{N}}$ be a collection of sets and let B be a set.
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- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- That is, $x \in \bigcap_{i\in\mathbb{N}} A_i$ and $x \notin B$.
- Therefore, $x \in \left(\bigcup_{i\in\mathbb{N}} A_i\right) \setminus B$.

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- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of sets and let B be a set.
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- So $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- That is, $x \in \bigcap_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- It follows that $\bigcup_{i \in \mathbb{N}} (A_i \setminus B) \subseteq \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

□

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
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- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.

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- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- It follows that $\left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B \subseteq \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.

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- Now suppose that $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- Then $x \in \bigcup_{i \in \mathbb{N}} A_i$ and $x \notin B$.
- That is, $x \in A_i$ for all $i \in \mathbb{N}$ and $x \notin B$.
- So $x \in A_i \setminus B$ for all $i \in \mathbb{N}$.
- Therefore, $x \in \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.
- It follows that $\left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B \subseteq \bigcup_{i \in \mathbb{N}} (A_i \setminus B)$.
- Thus, $\bigcup_{i \in \mathbb{N}} (A_i \setminus B) = \left(\bigcup_{i \in \mathbb{N}} A_i\right) \setminus B$.

□

Outline

Questions

- If $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$, what is $\bigcup_{i\in\mathbb{N}} A_i$?

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Questions

$\left(\bigcup_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcup_{j\in\mathbb{N}} B_j\right)$	$\bigcup_{i\in\mathbb{N}} \left(\bigcup_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$
$\left(\bigcup_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcap_{j\in\mathbb{N}} B_j\right)$	$\bigcup_{i\in\mathbb{N}} \left(\bigcap_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$
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$\left(\bigcap_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcap_{j\in\mathbb{N}} B_j\right)$	$\bigcap_{i\in\mathbb{N}} \left(\bigcap_{j\in\mathbb{N}}(A_i \setminus B_j)\right)$

Which sets are equal?

Outline

Axioms

Axiom

Let p and q be real numbers with $p < q$. Then there exists a real number r such that $p < r < q$.

Axiom

Let r be a real number. Then there exist two real numbers p and q such that $p < r < q$.

Axioms

Axiom

Let p, q, and r be real numbers. Then

(1) *(Transitivity) If $p < q$ and $q < r$, then $p < r$.*

(2) *(Linear ordering) Exactly one of the following statements is true:*

   (i) $p < q$

   (ii) $p = q$

   (iii) $p > q$

Axioms

Axiom (Archimedean Property)

Let x be a real number. Then there exists an integer n such that $n \leq x < n + 1$.

Outline

Open and Closed Intervals

Definition (Open Interval)

Let a and b be real numbers. The **open interval**, denoted (a, b), is the set

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

Definition (Closed Interval)

Let a and b be real numbers. The **closed interval**, denoted $[a, b]$, is the set

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Open and Closed Sets

Definition (Open Set)

A set S is **open** if, for every $x \in S$, there exist $a, b \in S$ such that $a < x < b$.

Definition (Closed Set)

A set S is **closed** if $\mathbb{R} \setminus S$ is open.

- Many sets are neither open nor closed.

Union and Intersection of Open Sets

Theorem

Let A and B be open sets. Then $A \cup B$ and $A \cap B$ are open sets.

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.

□

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.

$\square$

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.

□

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.
- Case 1: Suppose $x \in A$.

□

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.
- Case 1: Suppose $x \in A$.
  - Then there exists an interval $(a, b) \subseteq A$ with $x \in (a, b)$.

□

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.
- Case 1: Suppose $x \in A$.
  - Then there exists an interval $(a, b) \subseteq A$ with $x \in (a, b)$.
  - It follows that $(a, b) \subseteq A \cup B$.

□

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.
- Case 1: Suppose $x \in A$.
  - Then there exists an interval $(a, b) \subseteq A$ with $x \in (a, b)$.
  - It follows that $(a, b) \subseteq A \cup B$.
- Case 2: Suppose $x \in B$.

□

Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.
- Case 1: Suppose $x \in A$.
  - Then there exists an interval $(a, b) \subseteq A$ with $x \in (a, b)$.
  - It follows that $(a, b) \subseteq A \cup B$.
- Case 2: Suppose $x \in B$.
  - As in Case 1, we have $x \in (c, d) \subseteq A \cup B$ for some interval (c, d).

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Union and Intersection of Open Sets

Proof.

- Let A and B be open sets.
- Let $x \in A \cup B$.
- Then $x \in A$ or $x \in B$.
- Case 1: Suppose $x \in A$.
  - Then there exists an interval $(a, b) \subseteq A$ with $x \in (a, b)$.
  - It follows that $(a, b) \subseteq A \cup B$.
- Case 2: Suppose $x \in B$.
  - As in Case 1, we have $x \in (c, d) \subseteq A \cup B$ for some interval (c, d).
- Therefore, $A \cup B$ is an open set.

□

Outline

Homework Problem

Homework Problem 194(b)

Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a collection of closed sets. Prove that

$$\bigcap_{\alpha \in \Delta} U_\alpha$$

is a closed set.