Direct Proof – Introduction

Lecture 12
Section 4.1

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Mon, Feb 3, 2014

Outline

Proofs

- A proof is an argument leading from a hypothesis to a conclusion in which each step is so simple that its validity is beyond doubt.
- Simplicity is a subjective judgment – what is simple to one person may not be so simple to another.
- The writer of the proof must keep in mind his audience.

Statements to Prove or Disprove

- Proving universal statements – that something is true in every instance
- Proving existential statements – that something is true in at least one instance
- Disproving universal statements – that something is false in at least one instance
- Disproving existential statement – that something is false in every instance

Example

Theorem

For every real number a and for every real number $b > a$ there exists a real number c such that $a < c < b$.

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Outline

Proving Existential Statements

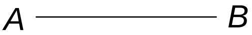
- Proofs of existential statements are often called existence proofs.
- Two types of existence proofs
  - Constructive – Find an instance where the statement is true.
  - Non-constructive – Argue indirectly that the there must be an instance where the statement is true.

Outline

Example

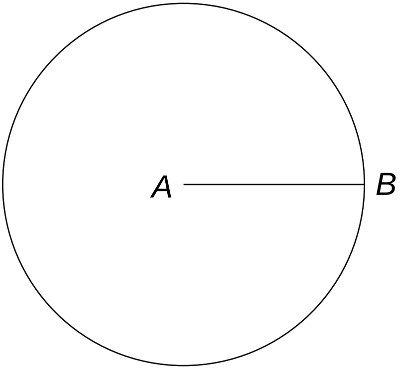
- Prove that every line segment AB has a midpoint M such that $AM = MB$.

Constructive Proof



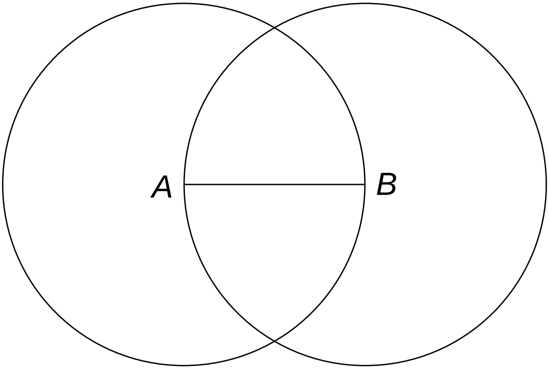
Given AB

Constructive Proof



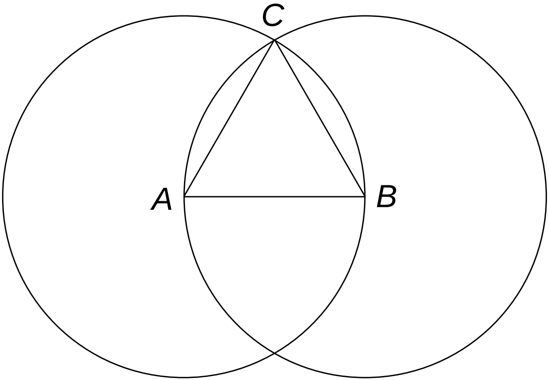
Draw circle with center A and radius AB

Constructive Proof



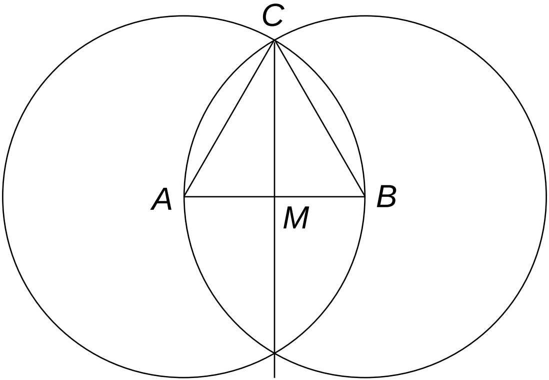
Draw circle with center B and radius BA

Constructive Proof



From intersection C draw equilateral triangle

Constructive Proof



Angle bisector at C bisects AB at M

Constructive Proof

- Now prove that the construction is correct.

Constructive Proof

Theorem

The equation

$$x^2 - 7y^2 = 1$$

has a solution in positive integers.

- Prove it by finding positive integers x and y that satisfy the equation.

Outline

Non-Constructive Proof

Theorem

The equation

$$x^5 - 3x + 1 = 0$$

has a solution in $\mathbb{R}$.

- Prove it by using continuity to argue indirectly that a solution must exist.

Outline

Negations of Universal Statements

- The negation of a universal statement is an existential statement.
- To prove the negation, construct an instance for which the statement is false or prove that one must exist.
- This is also called proof by counterexample.

Example

Theorem

The equation

$$\frac{x}{x+y} = \frac{1}{1+y}$$

does not hold for all real numbers x and y.

- Prove it by finding a counterexample.
- Is this a "constructive" proof? Of what statement?
- (For which real numbers does it hold?)

Outline

Odd and Even Integers

Definition (Odd and Even Integers)

An integer a is even if there exists an integer b such that $a = 2b$. Otherwise, a is odd.

- Is 0 even?
- Must every integer be either even or odd?
- Can an integer be both even and odd?
- How can we characterize odd integers in a positive way.

Example

Theorem

Let a, b, and c be integers. If a is even and $a = bc$, then b and c are even.

Theorem

Let a, b, and c be integers. If a is odd and $a = bc$, then b and c are odd.

- Is either “theorem” true?
- Prove whichever ones, if any, are true.

Outline

Collected

Collected

- Perform signed subtraction and check for overflow.

$$\begin{array}{r} 10010011 \\ -11011101 \\ \hline \end{array}$$

- Sec 3.1: 18cd, 25df.
- Sec 3.2: 17, 25.
- Sec 3.3: 10ef, 21cd.

Assignment

Assignment

- Read Section 4.1, pages 145 - 160.
- Exercises 3, 4, 8, 10, 12, 14, 18, 19, page 161.