Properties of Sets

Lecture 29
Section 6.2

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Outline

Properties of Sets

Commutative Laws	
Sets	Boolean
$A \cup B = B \cup A$	$p \vee q \equiv q \vee p$
$A \cap B = B \cap A$	$p \wedge q \equiv q \wedge p$

Associative Laws	
Sets	Boolean
$(A \cup B) \cup C = A \cup (B \cup C)$	$(p \vee q) \vee r) \equiv p \vee (q \vee r)$
$(A \cap B) \cap C = A \cap (B \cap C)$	$(p \wedge q) \wedge r) \equiv p \wedge (q \wedge r)$

Distributive Laws	
Sets	Boolean
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

Properties of Sets

Identity Laws	
Sets	Boolean
$A \cup \varnothing = A$ $A \cap U = A$	$p \vee F \equiv p$ $p \wedge T \equiv p$

Complement Laws	
Sets	Boolean
$A \cup A^c = U$ $A \cap A^c = \varnothing$	$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$

Double Complement Laws	
Sets	Boolean
$(A^c)^c = A$	$\sim (\sim p) \equiv p$

Properties of Sets

Idempotent Laws	
Sets	Boolean
$A \cup A = A$ $A \cap A = A$	$p \vee p \equiv p$ $p \wedge p \equiv p$

Universal Bound Laws	
Sets	Boolean
$A \cup U = U$ $A \cap \varnothing = \varnothing$	$p \vee T \equiv T$ $p \wedge F \equiv F$

DeMorgan's Laws	
Sets	Boolean
$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$ $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Properties of Sets

Absorption Laws	
Sets	Boolean
$A \cup (A \cap B) = A$	$p \vee (p \wedge q) \equiv p$
$A \cap (A \cup B) = A$	$p \wedge (p \vee q) \equiv p$

Complements of U and $\varnothing$	
Sets	Boolean
$U^c = \varnothing$	$\sim T \equiv F$
$\varnothing^c = U$	$\sim F \equiv T$

Examples

- The connection between sets and boolean expressions is the predicate.
- For example, recall

$$\begin{aligned} A \cup B &= \{x \mid P(x) \vee Q(x)\}, \\ A \cap B &= \{x \mid P(x) \wedge Q(x)\}, \\ A^c &= \{x \mid \sim P(x)\}. \end{aligned}$$

Example

- What set expressions are equivalent to

$$\begin{aligned} p &\to q, \\ p &\leftrightarrow q, \\ p &\oplus q, \\ p \text{ NAND } q&, \\ p \text{ NOR } q&. \end{aligned}$$

Outline

1. Properties of Sets
2. Proofs with Sets
3. Assignment

Proving Equality of Sets

- To prove that two sets A and B are equal,
  - Prove that $A \subseteq B$ and
  - Prove that $B \subseteq A$.

Example

Theorem

For any three sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example

Proof.

- Let A, B, C be any three sets.
- First, show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
  - Let $x \in A \cup (B \cap C)$.
  - Then $x \in A$ or $x \in B \cap C$.
  - If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$.
  - So $x \in (A \cup B) \cap (A \cup C)$.

$\square$

Example

Proof.

- On the other hand, if $x \in B \cap C$, then $x \in B$ and $x \in C$.
- So $x \in A \cup B$ and $x \in A \cup C$.
- Therefore, $x \in (A \cup B) \cap (A \cup C)$.

□

Example

Proof.

- Now, show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
  - Let $x \in (A \cup B) \cap (A \cup C)$.
  - Then $x \in A \cup B$ and $x \in A \cup C$.
  - If $x \in A$, then $x \in A \cup (B \cap C)$.
  - If $x \notin A$, then $x \in B$ and $x \in C$.
  - So $x \in B \cap C$ and therefore, $x \in A \cup (B \cap C)$.
- Therefore, $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

□

Proving a Set is Empty

- To prove that a set A is empty,
  - Let $x \in A$, i.e., suppose that $A \neq \varnothing$.
  - Reach a contradiction.

Example

Theorem

For any sets A and B,

$$(A - B) \cap (B - A) = \varnothing.$$

Example

Proof.

- Let A and B be sets and let $x \in (A - B) \cap (B - A)$.
- Then $x \in A - B$ and $x \in B - A$.
- But if $x \in A - B$, then $x \in A$ and $x \notin B$.
- And if $x \in B - A$, then $x \in B$ and $x \notin A$.
- This is a contradiction, so $(A - B) \cap (B - A) = \varnothing$.

□

Example

Theorem

For any sets A, B, and C, if $A \subseteq B$ and $B \cap C = \varnothing$, then $A \cap C = \varnothing$.

Example

Proof.

- Let A, B, and C be sets and let $x \in A \cap C$.
- Then $x \in A$ and $x \in C$.
- But if $x \in A$, then $x \in B$, because $A \subseteq B$.
- So $x \in B \cap C$.
- But $B \cap C = \varnothing$, which is a contradiction.
- Therefore, $A \cap C = \varnothing$.

□

Sets and Boolean Expressions

- Translate the last two theorems into boolean expressions.
- Do the proofs also translate?

Outline

Assignment

Assignment

- Read Section 6.2, pages 352 - 364.
- Exercises 3, 4, 9, 10, 14, 15, 18, 19, 31, 32, 34, page 364.