Functions and Relations

Lecture 31
Section 7.1

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Outline

Relations

Definition (Relation)

Let X and Y be sets. A relation R from X to Y is a subset of $X \times Y$.

- The idea is that if $(x, y) \in R$, then x "has the relation" to y.
- For example, if $X = \{2, 3, 4, 5\}$, $Y = \{1, 2, 3, 4\}$, and the relation R from X to Y is the "less than" relation, then

$$R = \{(2, 3), (2, 4), (3, 4)\}.$$

- Describe the "greater than" relation from X to Y.

Relations

- Each member of either set in the relation may have the relation to any number of elements in the other set.
- In other words, the relation R is a subset of $X \times Y$.
- In C++, some standard relations are $==$, $!=$, $<$, $>$, $<=$, and $>=$.
- We will study relations in more detail later.

Outline

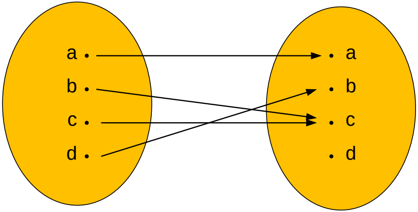
Function

Definition (Function)

A function f from X to Y, denoted $f : X \to Y$, is a relation from X to Y with the property that *for every* $x \in X$, there exists *exactly one* $y \in Y$ such that $(x, y) \in f$. Traditionally we write $f(x) = y$, although we can write $f : x \mapsto y$. X is the domain of f and Y is the codomain of f.

- An example is the function $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = x^2$.
- Some members of f are $(0, 0), (1, 1), (2, 4), (3, 9)$, and $(-3, 9)$.

Function Diagrams



- For very small sets, we may draw diagrams of functions.
- The above diagram shows a function

$$f : \{a, b, c, d\} \to \{a, b, c, d\}.$$

- It defines the function $\{(a, a), (b, c), (c, c), (d, b)\}$.

Images of Sets

- Let $f : X \to Y$ and $A \subseteq X$.
- Then we may define $f(A)$ to be

$$\begin{aligned} f(A) &= \{f(x) \mid x \in A\} \\ &= \{y \in Y \mid y = f(x) \text{ for some } x \in A\}. \end{aligned}$$

- $f(A)$ is called the image of A.
- Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$ and let $f : A \to \mathbb{Z}$ by $f(x) = x^2$.
- Then $f(A) = \{0, 1, 4, 9\}$.

Range of a Function

Definition (Range of a Function)

The range of a function $f : X \to Y$ is the set $f(X)$.

- The range is a subset of the codomain, but not necessarily equal to the codomain.
- Let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2$.
- Then the range of f is $\{0, 1, 4, 9, 16, \ldots\}$.

Inverse Images of Sets

- We may reverse the process.
- If $f : X \to Y$ and $B \subseteq Y$, then

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

- $f^{-1}(B)$ is called the inverse image of B.
- Let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2$.
- Then $f^{-1}(4) = \{-2, 2\}$ and $f^{-1}(2) = \varnothing$.

The Identity Function

Definition (Identity Function)

Given a set X, the identity function on X is the function $I_X : X \to X$ defined by $I_X(x) = x$ for all $x \in X$.

Outline

Functions and Relations

- A relation R from X to Y may not be a function, but it *induces* a function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$.
- Given R and a set $A \subseteq X$, define

$$f(A) = \{y \in Y \mid x \in A \text{ and } (x, y) \in R\}.$$

- In this manner, f maps *every* subset of X to *exactly one* subset of Y.

Example

- Let $X = \{2, 3, 4, 5\}$, $Y = \{6, 7, 8, 9\}$ and let R be the relation that $(x, y) \in R$ if x divides y.
- Describe the function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$.

Outline

1. Relations
2. Functions
3. Functions and Relations
4. **Well Defined Functions**
5. Assignment

Well Defined Functions

- If a "function" is defined carelessly, it may turn out not to be well defined and, thus, not a function.
- This problem typically arises when elements of the domain may be represented in more ways than one.
- For example, the rational number $\frac{1}{2}$ may also be represented as $\frac{2}{4}$ or $\frac{10}{20}$.
- Suppose that we define a function $f : \mathbb{Q} \to \mathbb{Q}$ by $f\left(\frac{a}{b}\right) = a + b$.
- Then $f\left(\frac{1}{2}\right) \neq f\left(\frac{2}{4}\right)$, so f is not well defined.

Well Defined Functions

- As another example, let $X = \mathbb{Q} - \{1\}$ and consider $f : X \to \mathbb{Q}$ by $f\left(\frac{a}{b}\right) = \frac{a+b}{a-b}$.
- Show that f is well defined.

Proof.

- Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q} - \{1\}$ and suppose that $\frac{a}{b} = \frac{c}{d}$.
- We must show that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.
- If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.
- Then (if $a \neq 0$)

$$f\left(\frac{a}{b}\right) = \frac{a+b}{a-b} = \frac{ac+bc}{ac-bc}$$
$$= \frac{ac+ad}{ac-ad}$$
$$= \frac{c+d}{c-d} = f\left(\frac{c}{d}\right).$$

- Therefore, f is well defined. (What if $a = 0$?)

□

Outline

Collected

Collected

- Sec. 5.8: 15, 17.
- Sec. 6.1: 6, 32b.
- Sec. 6.2: 9, 31.

Assignment

Assignment

- Read Sections 7.1, pages 383 - 393.
- Exercises 1, 4, 6, 9, 14, 27, 32, 33, 35, 38, 42, 47, 49, page 393.