Composition of Functions

Lecture 34
Section 7.3

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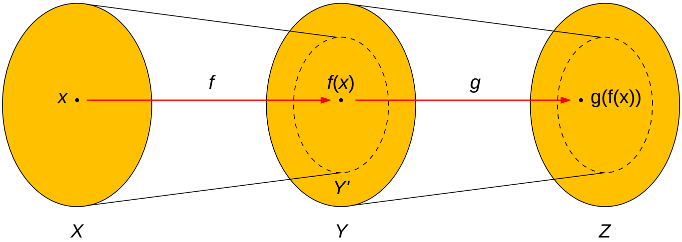
Outline

Composition of Functions

Definition (Composition of Functions)

Let $f : X \to Y$ and $g : Y' \to Z$, where range$(f) \subseteq Y'$. The composition of f with g is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.



Examples

- Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x + 3$.
- Let $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 4x + 5$.
- Find $f \circ g$ and $g \circ f$.
- Are they the same?

Composition with the Identity

- Recall that, for any set X, I_X is the identity function on X:

$$I_X(x) = x.$$

- Let $f : X \to Y$ be a function.
- Then

$$f \circ I_X = f$$

and

$$I_Y \circ f = f.$$

Outline

1. Composition of Functions
2. Inverses of Functions
3. Theorems about Composition
4. Assignment

Inverse of a Function

- Let $f : X \to Y$ be a one-to-one correspondence.
- Then the inverse $f^{-1} : Y \to X$ is a function and

$$f^{-1} \circ f = I_X$$

and

$$f \circ f^{-1} = I_Y.$$

Inverse of a Function

Theorem

Let $f : X \to Y$ *and* $g : Y \to X$. *Then* $g = f^{-1}$ *if and only if*

$$g \circ f = I_X$$

and

$$f \circ g = I_Y.$$

- This property allows us to *verify* that g is the inverse of f, but it does not show us how to *find* the inverse of f.

Example

- Let $f : \mathbb{R}^+ \to (1, \infty)$ by $f(x) = x^2 + 1$.
- Let $g : (1, \infty) \to \mathbb{R}^+$ by $g(x) = \sqrt{x - 1}$.
- Show that $g = f^{-1}$.

Exponents and Logarithms

- Let b be a real number with $b > 1$.
- Let $f : \mathbb{R} \to \mathbb{R}^+$ by $f(x) = b^x$.
- Let $g : \mathbb{R}^+ \to \mathbb{R}$ by $g(x) = \log_b x$.
- By definition,
$$\log_b b^x = x \text{ for all } x \in \mathbb{R}$$
and
$$b^{\log_b x} = x \text{ for all } x \in \mathbb{R}^+.$$
- Thus, f and g are inverses of each other.

Logarithms to Other Bases

- Suppose that we can find logarithms to the base b, but we want a logarithm to some other base a.
- For example, in Computer Science, we often need a base-2 logarithm, but our calculators have only base-10 and base-e logarithms.
- What do we do?

Logarithms to Other Bases

- Let $y = \log_a x$.
- Then

$$\begin{aligned} x &= a^y, \\ \log_b x &= \log_b a^y \\ &= y \log_b a. \end{aligned}$$

- Therefore,

$$y = \frac{\log_b x}{\log_b a}.$$

Example

- Find $\log_2 10$.
- Find $\log_2 100$.
- Find $\log_2 1000$.
- Find $\log_2 2000$.
- Find $\log_2 4000$.
- Find $\log_{16}$ of each of those 5 numbers.

Outline

1. Composition of Functions
2. Inverses of Functions
3. Theorems about Composition
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Composition of One-to-one Functions

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be one-to-one functions. Then $g \circ f : X \to Z$ is one-to-one. That is, the composition of one-to-one functions is one-to-one.

Composition of One-to-one Functions

Proof.

- Let $f : X \to Y$ and $g : Y \to Z$ be one-to-one functions.
- Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$ for some $x_1, x_2 \in X$.
- That is, $g(f(x_1)) = g(f(x_2))$.
- But g is one-to-one, so $f(x_1) = f(x_2)$.
- But then f is also one-to-one, so $x_1 = x_2$.
- Therefore, $g \circ f$ is one-to-one.

□

Composition of Onto Functions

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be onto functions. Then $g \circ f : X \to Z$ is onto. That is, the composition of onto functions is onto.

Composition of One-to-one Functions

Proof.

- Let $f : X \to Y$ and $g : Y \to Z$ be onto functions.
- Let $z \in Z$.
- Then, because g is onto, there exists $y \in Y$ such that $g(y) = z$.
- However, f is also onto, so there exists $x \in X$ such that $f(x) = y$.
- So $(g \circ f)(x) = g(f(x)) = g(y) = z$.
- Therefore, $g \circ f$ is onto.

□

Composition of Functions

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be functions and suppose that $g \circ f$ is one-to-one. Then f is one-to-one, but g is not necessarily one-to-one.

Composition of Functions

Proof.

- Let $f : X \to Y$ and $g : Y \to Z$ be functions and suppose that $g \circ f$ is one-to-one.
- Let $x_1, x_2 \in X$ and suppose that $f(x_1) = f(x_2)$.
- Then $g(f(x_1)) = g(f(x_2))$.
- It follows that $x_1 = x_2$ because $g \circ f$ is one-to-one.
- Therefore, f is one-to-one.

□

Composition of Functions

- How do we show that g is *not necessarily* one-to-one?
- All we need is one example where f is one-to-one and g is not one-to-one, but $g \circ f$ is one-to-one.
- Example?

Composition of Functions

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be functions and suppose that $g \circ f$ is onto. Then g is onto, but f is not necessarily onto.

Proof.

How would we prove this? □

Composition of Functions

Theorem

Let $f : X \to Y$ and $g : Y \to X$ be functions and suppose that $g \circ f = I_X$. Then f is one-to-one and g is onto.

Composition of Functions

Proof.

- The fact that f is one-to-one follows from an earlier theorem, because $g \circ f$ is one-to-one.
- The fact that g is onto follows from an earlier theorem, because $g \circ f$ is onto.

□

Examples

- Let $Y = \{x \in \mathbb{R} \mid x \geq 0\}$ and define
  - $f : \mathbb{R} \to Y$ by $f(x) = x^2$.
  - $g : Y \to \mathbb{R}$ by $g(x) = \sqrt{x}$.
- Is f one-to-one? onto?
- Is g one-to-one? onto?
- Is $g \circ f$ one-to-one? onto?
- Is $f \circ g$ one-to-one? onto?

Examples

- Let $X = \{x \in \mathbb{R} \mid x \geq 1\}$.
- Let $f : X \to \mathbb{R}$ by $f(x) = x \ln x$.
- Is f one-to-one? onto?
- What is f^{-1}?

Outline

Collected

Collected

- Sec. 7.1: 14, 42, 47.
- Sec. 7.2: 12b, 18, 49.

Assignment

Assignment

- Read Sections 7.3, pages 416 - 426.
- Exercises 1, 3, 8, 11, 16, 17, 21, 22, 24, page 426.